EECS 16B Designing Information Devices and Systems II
Fall 2019 UC Berkeley

Midterm 1

Exam location: NA

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ______________, ______________ ________________
(last) (first) (signature)

PRINT your discussion section and GSI (the one you attend): ________________________________

Row Number (front row is 1): ________ Seat Number (left most is 1): ________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

Section 0: Pre-exam questions (3 points)

1. What classes are you taking this term? (1 pt)

2. Tell us about something that makes you happy (2 pts)

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

3. High Speed Digital Communication (16 pts)

In this problem, we will analyze a simplified model of a USB communication link and show that there is a limit to how quickly you can transfer data.

As Figure 1 illustrates, the transmitter is a CMOS inverter, whose input is driven with the voltage source V_{in} representing the data to be sent over the link. The USB cable connects the transmitter to the receiver. The link successfully transfers data when the voltage at the output of the USB cable crosses the threshold of the receiver's inverter, thereby flipping the receiver output voltage $V_{out,Receiver}$.

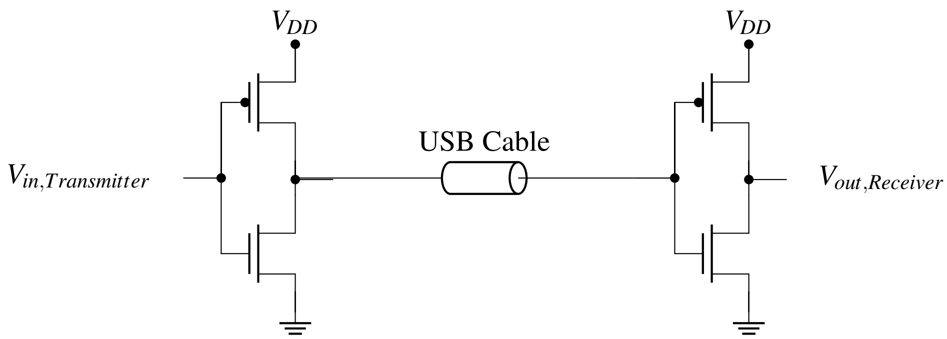


Figure 1: A USB communication link

To simplify, we will model the transmitter's inverter as a resistance R_{inv} and use $\bar{V}_{in}(t)$ as the digitally flipped voltage that represents the ideal output of the transmitter's inverter. The receiver inverter is modeled as an input capacitance C_{inv}. The cable is modeled as an RC system whose R_{wire} and C_{wire} values grow as the cable length increases. A diagram of the circuit model to be used in this problem is shown in Figure 3.

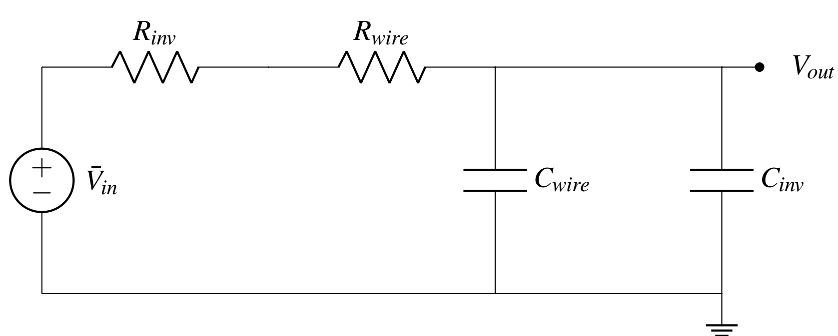


Figure 2: Simplified circuit model for a USB communication link

(a) (2 pts) **If $\bar{V}_{in} = 0V$ for all $t < 0$ what is V_{out} at time $t = 0$?** This will serve as the initial condition for the rest of the problem.

Solutions: Since $\bar{V}_{in} = 0V$ for a long time, we expect voltages and currents in the system to stop changing. If $\frac{dV_{out}}{dt} = 0$, that means the currents through the capacitors are zero. By KCL, that means that there is no current through the resistors, and so $V_{out} = \bar{V}_{in} = 0$

(b) (5 pts) **Write the differential equation for the voltage $V_{out}(t)$ as a function of $\bar{V}_{in}(t)$ for $t \geq 0$ in terms of $R_{inv}, C_{inv}, R_{wire}$, and C_{wire}.**

Solutions: Since the resistors are in series, they can be combined into a single resistance $R = R_{inv} + R_{wire}$. Since the capacitors are in parallel, they can be combined into a single capacitance $C = C_{inv} + C_{wire}$ We will use this simplified model for the rest of this solution.

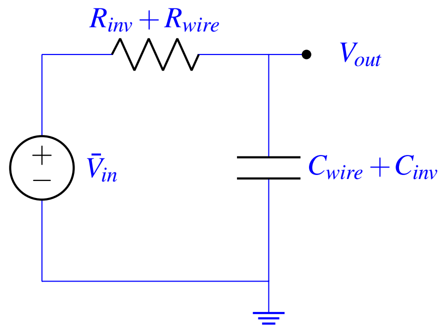


Figure 3: Simplified circuit model for a USB communication link

$$V_{in} = V_R + V_{out}$$

By KCL, the current through the resistor must equal the current through the capacitor:

$$V_R = I_C R = C\frac{dV_{out}}{dt}R$$

$$V_{in} = RC\frac{dV_{out}}{dt} + V_{out}$$

$$\frac{dV_{out}}{dt} = \frac{V_{in} - V_{out}}{RC}$$

$$\frac{dV_{out}}{dt} = \frac{V_{in} - V_{out}}{(R_{inv} + R_{wire})(C_{inv} + C_{wire})}$$

(c) (5 pts) For the rest of this problem, for ease of computation (since you don't have calculators), assume that $R_{inv} = 1\text{k}\Omega$, $R_{wire} = 3\text{k}\Omega$, $C_{wire} = 24\text{pF}$, and $C_{inv} = 1\text{pF}$. (Here, $\text{pF} = 10^{-12}\text{F}$ and $\text{k}\Omega = 10^3\Omega$.)

Now assume that $\bar{V}_{in}$ for time $t \geq 0$ is a piecewise-constant voltage source. $\bar{V}_{in}$ rises to 1V at $t = 0$ and then falls back down to 0V at some time $t = T_{bit}$. **Write an exact expression for $V_{out}(t)$ during the time period $0 \leq t < T_{bit}$.**

Your expression should be an explicit formula. No integrals are allowed to remain for full credit.

(HINT: Recall that $x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} u(\tau) d\tau$ is the unique solution of $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$ with initial condition $x(0) = x_0$. You also might want to draw $\bar{V}_{in}(t)$ to help yourself.)

Solutions: Since we know that the input is a constant $1V$ during the period $0 \leq t \leq T_{bit}$, we can simplify the differential equation:

$$\frac{dV_{out}}{dt} = -\frac{V_{out} - 1}{(R_{wire} + R_{inv})(C_{wire} + C_{inv})}$$

$$\frac{dV_{out}}{dt} = \lambda(V_{out} - 1)$$

Where, $\lambda = -\frac{1}{(R_{wire}+R_{inv})(C_{wire}+C_{inv})} = -10^7$

We can perform a change of variables to simplify the differential equation, $x(t) = V_{out}(t) - 1$

$$\frac{d}{dt}x(t) = \lambda x(t)$$

$$x(t) = x(0)e^{\lambda t}$$

Solving for the initial condition:

$$x(0) = V_{out}(0) - 1 = -1$$

Transforming back to standard variables:

$$V_{out}(t) = 1 + x(t) = 1 - e^{\lambda t} = 1 - e^{-10^7 t}$$

(d) (4 pts) **Sketch $V_{out}(t)$ during the time period $0 \leq t < T_{bit}$, clearly marking the initial voltage and the final voltage.**

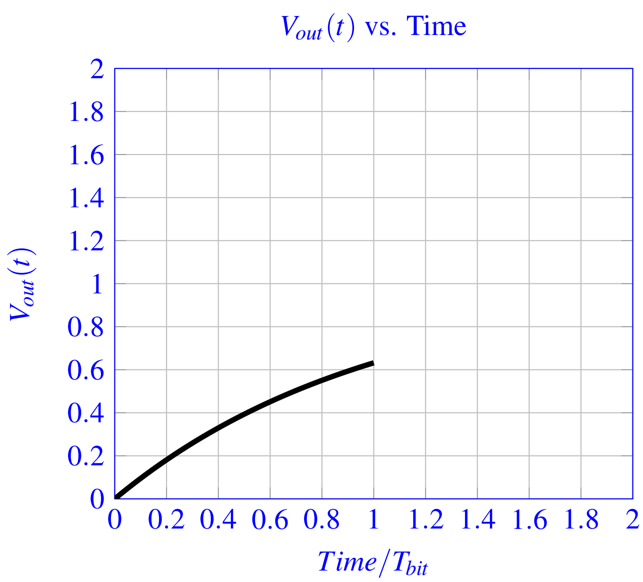
Solutions:

The plot below shows an example of $V_{out}(t)$ from 0 to T_{bit} in the case where $T_{bit} = RC$. Note that since we did not specify the value of T_{bit}, your plot may extend closer or farther from the asymptotic value of $1V$, as long as the relative shape and slope is consistent with $V_{out}(t) = 1 - e^{-10^7 t}$. The starting voltage of the plot must be consistent with the initial condition:

$$V_{out}(0) = 0V$$

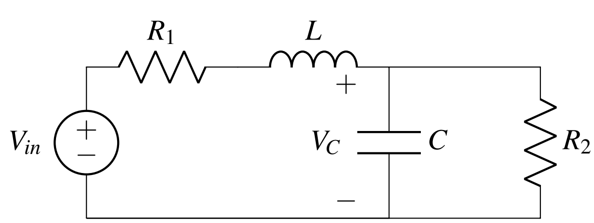
The final voltage of the plot can be computed in general by plugging T_{bit} into the expression from the previous part:

$$V_{out}(T_{bit}) = 1 - e^{-10^7 T_{bit}}$$



4. RLC Circuits (30 pts)

Consider the following circuit:



If we define the state vector $\vec{x}(t) = \begin{bmatrix} V_C \\ I_L \end{bmatrix}$, standard circuit analysis would reveal that this circuit is governed by this system of differential equations:

$$\frac{d}{dt}\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}}_{A} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\vec{b}} V_{in}(t).$$

For this problem generally, it may help to recall the standard formula for a 2×2 matrix inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(a) (12 pts) Suppose that we choose to drive the circuit with a $V_{in}(t)$ that is a sinusoidal waveform at the angular frequency ω radians/sec. Let us consider the voltage $V_C(t)$ across the capacitor C as our output voltage $V_{out}(t)$. **What is the transfer function of this circuit, namely** $H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$ where $\widetilde{V}_{in}$ is the input voltage phasor at angular frequency ω and $\widetilde{V}_{out}$ is the output voltage phasor at that same angular frequency ω.

Your answer should be symbolic in terms of the R_1, L, C, and R_2 along with $j = \sqrt{-1}$ and ω. You don't have to simplify this to look nice.

Solutions: We know that the transfer function is determined by an impedance voltage divider. With the equations $Z_R = R$, $Z_L = j\omega L$, $Z_C = \frac{1}{j\omega C}$, we get:

$$Z_{in}(j\omega) = Z_{R1} + Z_L + Z_{(C||R_2)} = R_1 + j\omega L + \frac{\frac{1}{j\omega C} R_2}{\frac{1}{j\omega C} + R_2}$$

Then simplifying a little for the next part:

$$Z_{in}(j\omega) = (R_1 + j\omega L) + \frac{R_2}{1 + j\omega C R_2} = \frac{(1 + j\omega C R_2)(R_1 + j\omega L) + R_2}{1 + j\omega C R_2}$$

Namely,

$$V_{out}(j\omega) = V_{in}(j\omega) \frac{\frac{R_2}{1+j\omega C R_2}}{\frac{(1+j\omega C R_2)(R_1+j\omega L)+R_2}{1+j\omega C R_2}}$$

$$H(j\omega) = \frac{V_{in}(j\omega)}{V_{out}(j\omega)} = \frac{R_2}{(1 + j\omega C R_2)(R_1 + j\omega L) + R_2}$$

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

(b) (8 pts) Assume that initially the state is at rest, with the capacitor charged to $1V$, so $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Consider the values $C = 1, L = 1, R_1 = 1, R_2 = 1$. For convenience, we have plugged in the values as well as computed eigenvalues and eigenvectors for you. This yields an A matrix with eigenvalues $\lambda_1 = -1 + j$ and $\lambda_2 = -1 - j$.

$$\frac{d}{dt}\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} V_{in}(t), \quad \text{and eigenvectors } \vec{v}_{\lambda=-1+j} = \begin{bmatrix} -j \\ 1 \end{bmatrix}, \vec{v}_{\lambda=-1-j} = \begin{bmatrix} j \\ 1 \end{bmatrix}$$

For convenience, please note that if $V = [\vec{v}_{\lambda=-1+j}, \vec{v}_{\lambda=-1-j}]$, then $V^{-1} = \begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{j}{2} & \frac{1}{2} \\ -\frac{j}{2} & \frac{1}{2} \end{bmatrix}$.

Change coordinates of this differential equation to be in terms of $\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$ and the input $V_{in}(t)$. i.e. give an equation in the form $\frac{d}{dt}\vec{\tilde{x}}(t) = \widetilde{A}\vec{\tilde{x}}(t) + \vec{\tilde{b}}V_{in}(t)$.

What is the matrix $\widetilde{A}$, the vector $\vec{\tilde{b}}$ and the intial condition $\vec{\tilde{x}}(0)$?

Solutions: Remember that $\widetilde{A} = V^{-1}AV$ is a matrix whose diagonal elements are the eigenvalues:

$$\widetilde{A} = \begin{bmatrix} -1+j & 0 \\ 0 & -1-j \end{bmatrix}$$

In order to convert b and $\vec{x}(0)$ to the eigenbasis, we need to multiply by V^{-1}:

$$\vec{\tilde{b}} = V^{-1}b$$

$$\vec{\tilde{b}} = \begin{bmatrix} \frac{j}{2} & \frac{1}{2} \\ -\frac{j}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{\tilde{b}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0)$$

$$\vec{\tilde{x}}(0) = \begin{bmatrix} \frac{j}{2} & \frac{1}{2} \\ -\frac{j}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{\tilde{x}}(0) = \begin{bmatrix} \frac{j}{2} \\ -\frac{j}{2} \end{bmatrix}$$

(c) (10 pts) For the problem as stated in the previous part, **solve for the transient solution $\vec{x}(t)$ given the specified initial condition under the assumption that $V_{in}(t) = 0$ for all $t \geq 0$.**

Your solution should be expressed in terms of real functions of time t. No j's are permitted in your final answer for full credit.

Solutions: Since $V_{in}(t) = 0$ for all $t \geq 0$, we can ignore the $\vec{\tilde{b}}V_{in}(t)$ term in the differential equation:

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \widetilde{A}\vec{\tilde{x}}(t)$$

Plugging in for $\widetilde{A}$ found in the previous part:

$$\frac{d}{dt}\widetilde{x}[0](t) = (-1+j)\widetilde{x}[0](t)$$

$$\frac{d}{dt}\widetilde{x}[1](t) = (-1-j)\widetilde{x}[1](t)$$

Each equation can be solved for independently:

$$\widetilde{x}[0](t) = \widetilde{x}[0](0)e^{(-1+j)t}$$

$$\widetilde{x}[1](t) = \widetilde{x}[1](0)e^{(-1-j)t}$$

Plugging in for the initial conditions found in the previous part:

$$\widetilde{x}[0](t) = \frac{j}{2}e^{(-1+j)t}$$

$$\widetilde{x}[1](t) = -\frac{j}{2}e^{(-1-j)t}$$

To convert back to the original basis $\vec{x}(t) = V\vec{\tilde{x}}(t)$:

$$\vec{x}(t) = \begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{j}{2}e^{(-1+j)t} \\ -\frac{j}{2}e^{(-1-j)t} \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} \frac{1}{2}e^{t}(e^{jt} + e^{-jt}) \\ \frac{j}{2}e^{-t}(e^{jt} - e^{-jt}) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} e^{-t}cos(t) \\ -e^{-t}sin(t) \end{bmatrix}$$

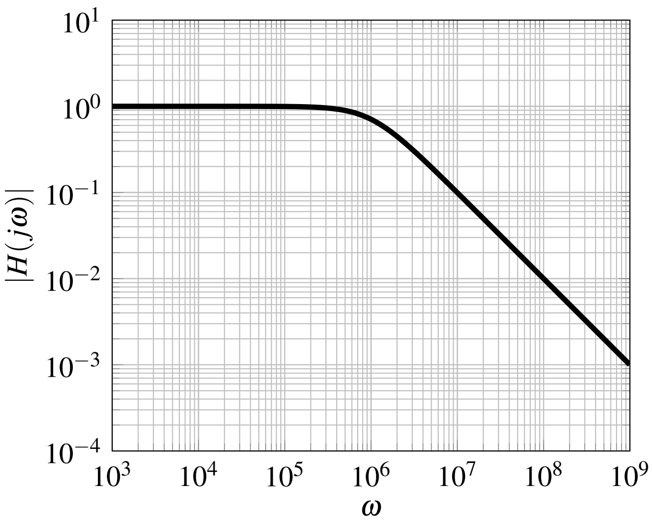
5. Transfer Functions and Filters (18 pts)

(a) (6 pts) **Identify each of the Bode Plots, circuits, and transfer functions as either a lowpass or highpass filter.** Indicate your answer by filling in the appropriate bubble.

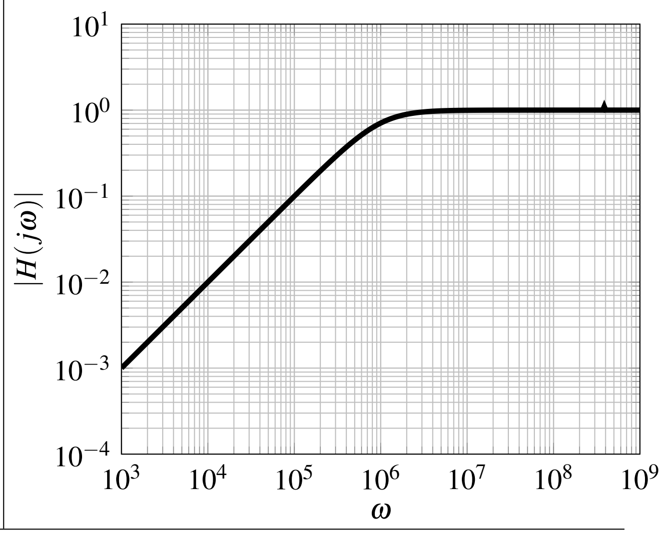
Table 1: Table to be filled in for your answers. Fill in bubbles.

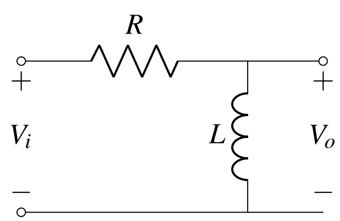
	Lowpass	Highpass
Bode Plot A	◯	◯
Bode Plot B	◯	◯
Circuit C	◯	◯
Circuit D	◯	◯
Transfer Fn E	◯	◯
Transfer Fn F	◯	◯

Bode Plot A ($\omega_c = 10^6$)

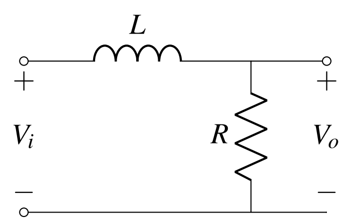


Bode Plot B ($\omega_c = 10^6$)





Circuit C: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$



Circuit D: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$

Transfer function E: $$H_E(j\omega) = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}}$$

Transfer function F: $$H_F(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_c}}$$

Solutions:

	Lowpass	Highpass
Graph	A	B
Circuit	D	C
Equation	F	E

(b) (6 pts) Consider the three filters in cascade below, with unity-gain op-amp buffers in between them:

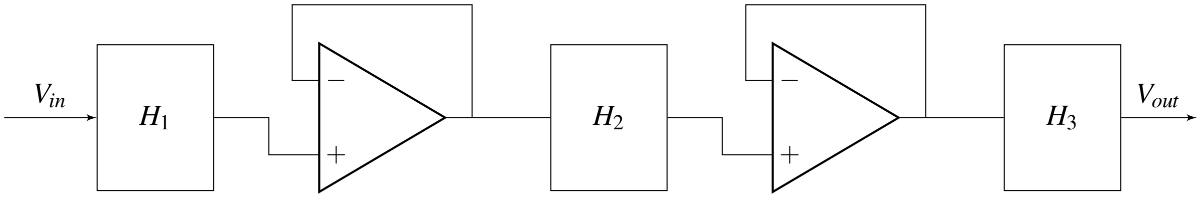


Figure 5: Three filters cascaded via unity-gain op-amp buffers

Suppose that at some frequency ω_0 radians/sec we know that:

$$H_1(j\omega_0) = 3e^{j\frac{\pi}{4}} \quad H_2(j\omega_0) = \frac{1}{2}e^{-j\frac{\pi}{3}} \quad H_3(j\omega_0) = 4e^{j\frac{5\pi}{6}}$$

If $V_{in}(t) = 2\sin\left(\omega_0 t + \frac{\pi}{2}\right)$:

What is the phasor for the input voltage: $\widetilde{V_{in}}$?

What is the phasor for the output voltage: $\widetilde{V_{out}}$?

What is $V_{out}(t)$?

Solutions: First we find the phasor representation of $V_{in}(t)$. Remember that $sin(t + \frac{\pi}{2}) = cos(t)$:

$$V_{in}(t) = 2cos(\omega_0 t)$$

$$\widetilde{V_{in}} = 1e^{j0} = 1$$

Then to get the overall transfer function, multply all the individual transfer functions. This is the same as multplying the phasor magnitudes and adding the phases:

$$H_{total} = \frac{V_{out}}{V_{in}} = H_1 \cdot H_2 \cdot H_3 = 6e^{j\frac{3\pi}{4}}$$

From phasor analysis, $\widetilde{V_{out}} = H_{total}\widetilde{V_{in}} = 6e^{j\frac{3\pi}{4}}$

Finally we convert back to the original time domain:

$$V_{out}(t) = 12cos(\omega_0 t + \frac{3\pi}{4})$$

This can be done instantly since 12 is twice of 6. And the phase shift $\frac{3\pi}{4}$ is the phase of the output voltage phasor. We know this from the equation for cosine: $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$.

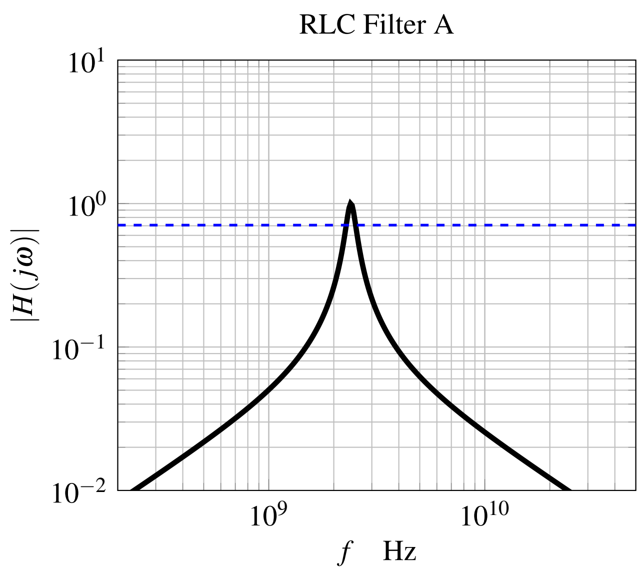
(c) (6 pts) Suppose there is an interfering signal at 5GHz that you need to get rid of, while passing through your WiFi signal at 2.4GHz. You have access to the following six components (two capacitors, two inductors, and two resistors) which should each only be used exactly once.

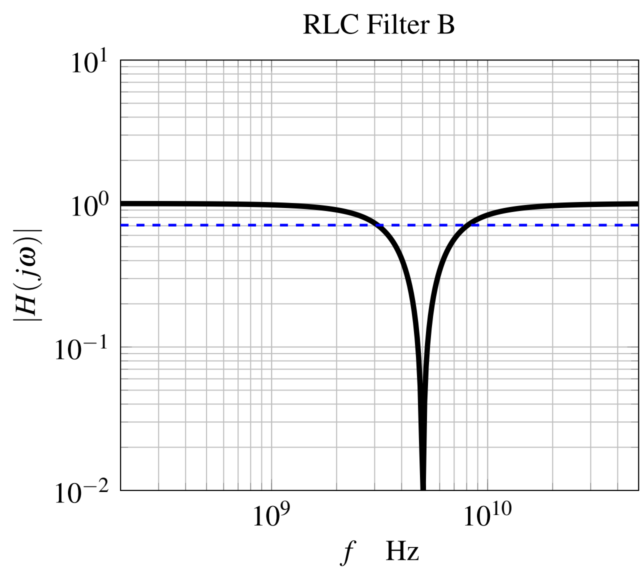
$$C = 66.3\text{pF} \quad C = 31.8\text{pF} \quad L = 31.8\text{pH} \quad L = 66.3\text{pH} \quad R = 1\Omega \quad R = 0.1\Omega$$

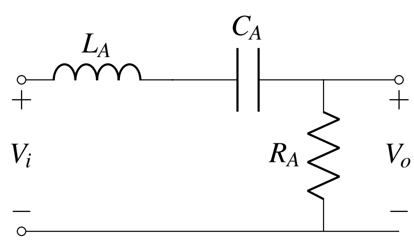
Assign each component to the elements R_A, R_B, C_A, C_B, L_A, L_B in the RLC circuits ('A' and 'B' below), so that the transfer function of each circuit corresponds to its matching Bode plot. Write values next to components. *Hint: the dashed line on the plot is at $\frac{1}{\sqrt{2}}$. It might be useful to think about what its intersections with the main curve represent.*

For your convenience, here are some calculations that may or may not be relevant:

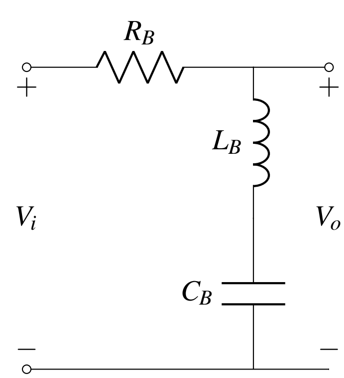
$\frac{2.4\times10^9}{2\pi} = 382\times10^6$	$2.4\times10^9\times2\pi = 15.08\times10^9$	$\frac{5.0\times10^9}{2\pi} = 795\times10^6$	$5.0\times10^9\times2\pi = 31.4\times10^9$
$\frac{1}{382\times10^6} = 2.62\times10^{-9}$	$\frac{1}{15.08\times10^9} = 6.63\times10^{-11}$	$\frac{1}{795\times10^6} = 1.26\times10^{-9}$	$\frac{1}{31.4\times10^9} = 3.18\times10^{-11}$







Circuit A: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$



Circuit B: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$

Solutions: Based on the transfer function plots, RLC Filter A is a band-pass filter centered around $2.4GHz$ and RLC Filter B is a notch filter centered around $5GHz$.

For the band-pass filter, the peak is located at the frequency where the sum of the impedances of the inductor and capacitor becomes 0. The same is true for the center of the notch-filter.

$$Z_L + Z_C = 0$$

$$j\omega_n L - \frac{j}{\omega_n C} = 0$$

$$\omega_n^2 L = \frac{1}{C}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

Since the notch frequency is slightly more than 2x the band-pass frequency, the $\sqrt{LC}$ value of the band-pass filter must be around 2x bigger than that of the band pass filter. Thus we will choose the pair of larger L and C values for the band-pass filter, each of which are each roughly 2x larger: $L_A = 66.3pH$ and $C_A = 66.3pF$. This can be confirmed by using the provided calculations. Since each component can only be used once, the notch filter must use: $L_B = 31.8pH$ and $C_B = 31.8pF$

From looking at the dotted lines on the plots, it is clear that the notch filter has much larger bandwidth than the band-pass filter. Lets evaulate the transfer function of the band-pass filter to determine the relationship between resistance and the cutoff frequencies. From the voltage divider formula:

$$H(j\omega) = \frac{R_A}{R_A + j(\omega L - \frac{1}{\omega C})}$$

The magnitude of this transfer function will hit $\frac{1}{\sqrt{2}}$ when the magnitude of the imaginary component of the denominator equals R_A. Clearly, a larger R_A leads to larger cutoff frequency. Alternatively, you could recall that the bandwidth of a band-pass filter is $\sqrt{\frac{R}{L}}$. Since the band-pass filter has a much smaller bandwidth than the notch-filter, we will choose the smaller resistance $R_A = 0.1\Omega$ and $R_B = 1\Omega$

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

6. Separation of Variables and Uniqueness (20 pts)

Recall that the classic scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

with initial condition $x(0) = x_0 \neq 0$ has the unique solution $x(T) = x_0 e^{\lambda T}$ for all $T \geq 0$.

(Note: to avoid variable-name confusion here, we are using T as the argument of the solution $x(T)$.)

The separation of variables approach to getting a guess for this problem would proceed as follows:

$$\begin{align}
\frac{d}{dt}x(t) &= \lambda x(t) \tag{2}\\
\frac{dx}{dt} &= \lambda x \tag{3}\\
\frac{dx}{x} &= \lambda dt \text{ separating variables to sides} \tag{4}\\
\int_{x_0}^{x(T)} \frac{dx}{x} &= \int_0^T \lambda dt \text{ integrating both sides from where they start to where they end up} \tag{5}\\
\ln(x(T)) - \ln(x_0) &= \lambda T \tag{6}\\
\ln(x(T)) &= \ln(x_0) + \lambda T \tag{7}\\
x(T) &= x_0 e^{\lambda T} \text{ exponentiating both sides} \tag{8}
\end{align}$$

and in this case it gave a good guess. Of course, this guess needed to be justified by a uniqueness proof, which you did in the homework.

This exam problem asks you to carry out this program for the time-varying differential equation:

$$\frac{d}{dt}x(t) = \lambda(t)x(t) \tag{9}$$

with initial condition $x(0) = x_0 \neq 0$. You can assume that $\lambda(t)$ is a nice continuously differentiable function of time t that is bounded.

(a) (8 pts) Use the separation of variables approach to get a guess for the solution to the differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with initial condition $x(0) = x_0 \neq 0$. **Show work and give a formula for $x(T)$ for $T \geq 0$.**

(HINT: It is fine if your answer involves a definite integral.)

(If you can't solve this for a general $\lambda(t)$, for partial credit, feel free to just consider the special case of $\lambda(t) = -2 - \sin(t)$ and give a guess for that case.)

(You can also get full credit if you follow the approach from discussion section of taking a piecewise-constant approximation and then taking a limit, but that might involve more work.)

Solutions:

$$\frac{d}{dt}x(t) = \lambda(t)x(t) \tag{10}$$

$$\frac{dx}{dt} = \lambda(t)x \tag{11}$$

$$\frac{dx}{x} = \lambda(t)dt \text{ separating variables to sides} \tag{12}$$

$$\int_{x_0}^{x(T)} \frac{dx}{x} = \int_0^T \lambda(t)dt \text{ integrating both sides from where they start to where they end up} \tag{13}$$

$$\ln(x(T)) - \ln(x_0) = \int_0^T \lambda(t)dt \tag{14}$$

$$\ln(x(T)) = \ln(x_0) + \int_0^T \lambda(t)dt \tag{15}$$

$$x(T) = x_0 e^{\int_0^T \lambda(t)dt} \text{ exponentiating both sides and folding constants} \tag{16}$$

which is a pretty reasonable guess since it definitely agrees with the correct answer if $\lambda(t)$ was just a constant.

For the special case of $\lambda(t) = -2 - \sin(t)$, we know that $\int_0^T \lambda(t)dt = \int_0^T -2 - \sin(t)dt = -2T + \cos(T) - 1$ and so $x(T) = x_0 e^{-2T+\cos(T)-1}$ is our guessed solution.

It turns out that the same limiting argument invoking piecewise-constants and Reimann sums that was done in discussion would also have resulted in the same guess.

(b) (12 pts) **Prove the uniqueness of the solution — i.e. that if any function solves differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with the given initial condition $x(0) = x_0 \neq 0$, then it must in fact be the same as your guessed solution everywhere for $T \geq 0$.**

(HINT: A ratio-based argument might be useful. You don't actually need to know the exact form of your guessed solution to carry out much of this argument, but you do need the fact that it is never zero and that it solves (9).)

Solutions: First, we need to know how our guessed solution behaves.

Plugging in the intial condition into $x(T) = x_0 e^{\int_0^T \lambda(t)dt}$, we get $x(0) = x_0 e^{\int_0^0 \lambda(t)dt}$, we get $x(0) = x_0 e^0 = x_0$.

Before plugging into (9), we first write the solution using t instead of T and changing the dummy variable for the integral to be τ: $x(t) = x_0 e^{\int_0^t \lambda(\tau)d\tau}$.

Plugging into (9), we get

$$\frac{d}{dt}x(t) = \frac{d}{dt}x_0 e^{\int_0^t \lambda(\tau)d\tau} \tag{17}$$

$$= x_0 \lambda(t) e^{\int_0^t \lambda(\tau)d\tau} \tag{18}$$

$$= \lambda(t)x(t) \tag{19}$$

which satisfies the differential equation.

Solutions: Consider any candidate solution $y(t)$ to (9) that satisfies the given initial condition $y(0) = x_0 \neq 0$. This means that $\frac{d}{dt}y(t) = \lambda(t)y(t)$ for all $t \geq 0$.

We know that our guessed solution $x(t) = x_0 e^{\int_0^t \lambda(\tau)d\tau}$ is never zero because $x_0 \neq 0$ and the finite integral of a bounded function cannot be $-\infty$. This means that we are free to consider the ratio $z(t) = \frac{y(t)}{x(t)}$. We know that $z(0) = \frac{y(0)}{x(0)} = \frac{x_0}{x_0} = 1$.

Looking at the derivative of $z(t)$, we have:

$$\frac{d}{dt}z(t) = \frac{d}{dt}\frac{y(t)}{x(t)} \tag{20}$$
$$= \frac{(\frac{d}{dt}y(t))x(t) - y(t)\frac{d}{dt}x(t)}{(x(t))^2} \tag{21}$$
$$= \frac{\lambda(t)y(t)x(t) - y(t)\lambda(t)x(t)}{(x(t))^2} \tag{22}$$
$$= 0. \tag{23}$$

Since $\frac{d}{dt}z(t) = 0$ for all $t \geq 0$, it is not changing, and is therefore a constant. This means that $z(t) = z(0) = 1$ which implies that $\frac{y(t)}{x(t)} = 1$ which means that $y(t) = x(t)$, thereby establishing uniqueness for our guessed solution.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]