Solutions to the Midterm Exam – Linear Algebra

Math 110, Fall 2019. Instructor: E. Frenkel

Problem 1. Let V be the subspace of $P_2(\mathbb{R})$ that consists of all polynomials $p(t)$ of degree less than or equal to 2, such that

$$\int_0^1 p(t)dt = 0.$$

Construct a basis β of V and prove that it is a basis.

Solution. Let $p(t) = a_0 + a_1 t + a_2 t^2$. Then $\int_0^1 p(t)dt = 0$ means that $a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = 0$. We claim that

$$\beta = \{1 - 2t, 1 - 3t^2\}$$

is a basis of this subspace (of course, it's just one of many possibilities). To prove this, note that this subspace – denote it by V – is the null-space $N(T)$ of the linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}$ sending $p(t)$ to $\int_0^1 p(t)dt$. This linear transformation is onto, because $\int_0^1 cdt = c$ for any $c \in \mathbb{R}$. Hence $R(T) = \mathbb{R}$, and by Dimension Theorem, $\dim V = 3-1 = 2$. Since β consists of two elements, in order to prove that β is a basis of V, it is sufficient to prove that β is linearly independent. Clearly, any non-zero scalar multiple of $(1-2t)$ is a polynomial of degree 1, so it cannot be equal to $(1-3t^2)$ which is a polynomial of degree 2. Therefore β is β is indeed linearly independent; hence a basis of V.

Problem 2. Let $M \in M_{n\times n}(F)$, where F is a field, be an upper triangular matrix with non-zero diagonal entries. Prove that the columns of M form a basis of F^n.

Solution. This was explained in detail during a lecture, and there was also a closely related homework problem.

We know that $\dim F^n = n$ (because it has a canonical basis with n elements). Since we have a set of n columns of M, if we prove that this set is linearly independent, then it will follow that it is a basis of F^n.

Denote the ith column by v_i. Suppose that we have a linear relation

$$\sum_{i=1}^{n} a_i v_i = \underline{0}, \qquad a_i \in F. \tag{1}$$

Suppose that at least one of the a_i is non-zero. Let j be the maximal integer from 1 to n such that $a_j \neq 0$. Then the jth entry of the LHS of (1) is equal to $a_j \cdot v_{jj}$, where v_{jj} is the jth entry of v_j, which is the diagonal entry M_{jj} of M. Since the diagonal entries of M are non-zero, we have $v_{jj} \neq 0$. We have assumed that $a_j \neq 0$. Hence $a_j \cdot v_{jj} \neq 0$, which means that the equation (1) cannot be satisfied. This is a contradiction. Hence all a_i are equal to 0, and the set $\{v_1, \dots, v_n\}$ is linearly independent. Therefore it is a basis of F^n.

Problem 3. Let $T : P_3(\mathbb{C}) \to P_2(\mathbb{C})$ be defined by the formula $T(p(t)) = 2p'(t) - 3p''(t)$.

Consider $P_3(\mathbb{C})$ and $P_2(\mathbb{C})$ as vector spaces over $\mathbb{C}$. Prove that T is a linear transformation between them and compute its matrix $[T]_\beta^\gamma$, where β is the standard monomial basis and $\gamma = \{1, t-1, t^2-1\}$.

Solution. We find

$$T(1) = 0, \qquad T(t) = 2 \cdot 1, \qquad T(t^2) = 4t - 6 = (-2) \cdot 1 + 4 \cdot (t-1),$$
$$T(t^3) = 6t^2 - 18t = (-12) \cdot 1 + 6 \cdot (t^2 - 1) - 18 \cdot (t-1)$$

Thus,

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 2 & -2 & -12 \\ 0 & 0 & 4 & -18 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Problem 4. Let V be a two-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear transformation. Suppose that $\beta = \{x_1, x_2\}$ and $\gamma = \{y_1, y_2\}$ are two bases in V such that

$$y_1 = x_1 + x_2, \qquad y_2 = x_1 + 2x_2.$$

Find $[T]_\beta$ if

$$[T]_\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

Solution. We have

$$[T]_\gamma = Q^{-1}[T]_\beta Q,$$

where

$$Q = \big([y_1]_\beta [y_2]_\beta\big) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

It is easy to compute that

$$Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Therefore

$$[T]_\beta = Q[T]_\gamma Q^{-1} = \begin{pmatrix} 10 & -5 \\ 15 & -7 \end{pmatrix}$$

Problem 5. Let $P_n^k(\mathbb{R})$ be the set of real polynomials $p(t)$ in one variable of degree less than or equal to n and such that the values of $p(t)$ at $t = 1, 2, \ldots, k$ are all equal to 0, i.e. $p(1) = p(2) = \ldots = p(k) = 0$. Assume that $0 < k \leq n$. Prove that $P_n^k(\mathbb{R})$ is a vector space over $\mathbb{R}$, and prove that the dimension of $P_n^k(\mathbb{R})$ is $n - k + 1$.

Solution. First, let's prove that $P_n^k(\mathbb{R})$ is a subspace of $P_n(\mathbb{R})$. By a theorem from the book, it is sufficient to show that $P_n^k(\mathbb{R})$ is closed under addition and scalar multiplication and that the zero polynomial is an element of $P_n^k(\mathbb{R})$. All three properties are clear. So $P_n^k(\mathbb{R})$ is a subspace of $P_n(\mathbb{R})$ and hence it is a vector space.

Now we compute the dimension of $P_n^k(\mathbb{R})$.

First computation. It is known from high school algebra that every polynomial $p(t)$ that vanishes at $c_1, \ldots, c_k$ has the form $p(t) = q(t) \prod_{i=1}^k (t - c_i)$, where $q(t)$ is another polynomial. Therefore every $p(t) \in P_n^k(\mathbb{R})$ has the form $q(t) \prod_{i=1}^k (t - i)$, where $q(t) \in P_{n-k}(\mathbb{R})$. Define a map $U : P_k(\mathbb{R}) \to P_n^k(\mathbb{R})$ sending $q(t)$ to $q(t) \prod_{i=1}^k (t - i)$. It is clear from the definition that U is a linear transformation, and furthermore, an isomorphism. Hence $\dim P_n^k(\mathbb{R}) = \dim P_{n-k}(\mathbb{R}) = n - k + 1$.

Second computation. Consider the map $T : P_n(\mathbb{R}) \to \mathbb{R}^k$ sending

$$p(t) \mapsto \begin{pmatrix} p(1) \\ p(2) \\ \dots \\ p(k) \end{pmatrix}$$

This is a linear transformation because the value of $cp(t) + q(t)$ at m is $cp(m) + q(m)$. Clearly, $N(T) = P_n^k(\mathbb{R})$, and we know that $\dim P_n(\mathbb{R}) = n + 1$. Hence, by Dimension Theorem, $\dim P_n^k(\mathbb{R}) = (n+1) - \dim R(T)$. To prove that that the dimension of $P_n^k(\mathbb{R})$ is $n - k + 1$, we therefore need to prove that T is onto.

This follows from the statement of homework problem 2.6.10(b): there exist polynomials $p_i(t), i = 1, \dots, n+1$, such that $p_i(j) = \delta_{i,j}$ for all $j = 1, \dots, n+1$. This means that

$$T\left(\sum_{i=1}^{k} a_i p_i(t)\right) = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_k \end{pmatrix}, \qquad \forall a_1, \dots, a_k \in \mathbb{R}.$$

Problem 6. Consider the vector space $W = \{p(t) = a + bt^2 | a, b \in \mathbb{R}\}$. Let f_1 and f_2 be the linear functionals on W, such that $f_1[p(t)] = p(1)$, and $f_2[p(t)] = p(2)$.

Find the basis of W for which $\{f_1, f_2\}$ is the dual basis.

Solution is similar to the solution of the homework problem 2.6.5 (which was explained during a lecture) and Example 4 of Section 2.6.

Problem 7. Let A and B be two $n \times n$ matrices such that $AB = I_n$. Prove that then necessarily $BA = I_n$ as well.

Solution. This was explained in detail during a lecture, and this was the homework problem 2.4.10. Let L_A (resp. L_B) be the linear transformation $F^n \to F^n$ sending $v \mapsto Av$ (resp. Bv). Then L_A (resp. L_B) is invertible if and only if A (resp. B) is invertible. Furthermore, $AB = I_n$ implies that $L_A \circ L_B = I_{F^n}$, hence invertible. But then $N(L_B) = \{\underline{0}\}$, for otherwise there is $v \neq \underline{0}$ such that $L_B(v) = \underline{0}$, and then $L_A \circ L_B(v) = L_A(L_B(v)) = L_A(\underline{0}) = \underline{0}$, which contradicts $L_A \circ L_B$ being invertible. Since $N(L_B) = \{\underline{0}\}$, L_B is one-to one. By the Dimension Theorem, $\dim R(L_B) = n$ and so L_B is also onto. Thus, L_B is invertible. Hence there exists a matrix C such that $CB = BC = I_n$. Now, multiplying both sides of $AB = I_n$ on the right by C we find that $(AB)C = C$, hence $A = AI_n = A(BC) = (AB)C = C$, and then $BC = I_n$ implies $BA = I_n$.

Remark. Note that it is necessary to prove first that B is invertible, i.e. there exists a matrix C such that $CB = BC = I_n$. Otherwise, there is no such thing as B^{-1}. Alternatively, one can prove that A is invertible and then use A^{-1} in a similar way. Otherwise, there is no such thing as A^{-1}. So, without either of these arguments, we cannot use A^{-1} or B^{-1}.

Recall that a matrix A is called invertible if $AB = I_n$ *and* $BA = I_n$. Formula $AB = I_n$ alone does not guarantee that A or B is invertible. For this reason, any solution to this problem in which the existence of A^{-1} or B^{-1} was taken for granted was given 0 points.

Alternative solution. Multiplying both sides of $AB = I_n$ on the left by B, we get $BAB = B$. Hence $(BA - I_n)B = 0$. Next, we prove (as above) that B is invertible. Then we claim that $BA - I_n = 0$, or equivalently, $(BA - I_n)x = 0$ for all $x \in F^n$. Indeed, since B is invertible, there exists $y \in F^n$ such that $x = By$. Hence $(BA - I_n)x = (BA - I_n)B \cdot y = 0 \cdot y = \underline{0}$. Thus, $BA - I_n = 0$, and so $BA = I_n$.

Remark. A number of students claimed that $(BA - I_n)B = 0$ implies $BA - I_n$. But this only follows if we prove first that B is invertible (see above). Note that if we have two $n \times n$ matrices X and Y, with $n > 1$, then $XY = 0$ does *not* imply that $X = 0$ or $Y = 0$.