## Name:

## SID:

## Name and SID of student to your left:

## Name and SID of student to your right:

## Exam Room:

○ Evans 60 ○ Kroeber 160 ○ Latimer 120 ○ North Gate 105
○ Pimentel 1 ○ Cory 293 ○ Soda 320 ○ Soda 380
○ Other

## Assigned Seat:

*Rules and Guidelines*

- **The exam has 16 pages, is out of 110 points, and will last 110 minutes.**
- Answer all questions. Read them carefully first. Not all parts of a problem are weighted equally.
- Write your student ID number in the indicated area on each page.
- Be precise and concise.
- When there is a box for an answer, **only the work in box provided will be graded.**
- You may use the blank page on the back for scratch work, but it will not be graded. Box numerical final answers.
- **The problems do not necessarily follow the order of increasing difficulty.** ***Avoid getting stuck on a problem.***
- Any algorithm covered in lecture can be used as a blackbox. Algorithms from homework need to be accompanied by a proof or justification as specified in the problem.
- You may assume that comparison of integers or real numbers, and addition, subtraction, multiplication and division of integers or real or complex numbers, require $O(1)$ time.
- There are warmup questions on the back page of the exam for while you wait.
- Good luck!

This page is deliberately blank. You may use it to report cheating incidents. Otherwise, we will not look at it.

## Discussion Section

Which of these do you consider to be your primary discussion section(s)? Feel free to choose multiple, or to select the last option if you do not attend a section. **Please color the checkbox completely. Do not just tick or cross the boxes.**

- ☐ Arpita, Thursday 9 - 10 am, Dwinelle 223
- ☐ Avni, Thursday 10 - 11 am, Dwinelle 215
- ☐ Emaan, Thursday 10 - 11 am, Etcheverry 3107
- ☐ Lynn, Thursday 11 - 12 pm, Barrows 118
- ☐ Dee, Thursday 11 - 12 pm, Wheeler 30
- ☐ Max, Thursday 12 - 1 pm, Wheeler 220
- ☐ Sean, Thursday 1 - 2 pm, Etcheverry 3105
- ☐ Jiazheng, Thursday 1 - 2 pm, Barrows 175
- ☐ Neha, Thursday 2 - 3 pm, Dwinelle 242
- ☐ Julia, Thursday 2 - 3 pm, Etcheverry 3105
- ☐ Henry, Thursday 3 - 4 pm, Haviland 12
- ☐ Kedar, Thursday 3 - 4 pm, Barrows 104
- ☐ Ajay, Thursday 4 - 5 pm, Barrows 136
- ☐ Varun, Thursday 4 - 5 pm, Dwinelle 242
- ☐ Gillian, Friday 9 - 10 am, Dwinelle 79
- ☐ Hermish, Friday 10 - 11 am, Evans 9
- ☐ Vishnu, Friday 11 - 12 pm, Wheeler 222
- ☐ Carlo, Friday 11 - 12 pm, Dwinelle 109
- ☐ Tarun, Friday 12 - 1 pm, Hildebrand B56
- ☐ Noah, Friday 12 - 1 pm, Hildebrand B51
- ☐ Teddy, Friday 1 - 2 pm, Dwinelle 105
- ☐ Jialin, Friday 1 - 2 pm, Wheeler 30
- ☐ Claire, Friday 2 - 3 pm, Barrows 155
- ☐ Jierui, Friday 2 - 3 pm, Wheeler 202
- ☐ David, Friday 2 - 3 pm, Wheeler 130
- ☐ Nate, Friday 2 - 3 pm, Evans 9
- ☐ Rishi, Friday 3 - 4 pm, Dwinelle 243
- ☐ Tiffany, Friday 3 - 4 pm, Barrows 104
- ☐ Ida, Friday 3 - 4 pm, Dwinelle 109
- ☐ Don't attend Section.

# 1 Asymptotics. (11 points.)

1. (**5 points, 1 point each.**) For each row, select the most specific asymptotic statement(s).

| $f(n)$ | $g(n)$ | $f(n) = O(g(n))$ | $f(n) = \Omega(g(n))$ | $f(n) = \Theta(g(n))$ |
|---|---|---|---|---|
| $n^2$ | $1000n + \log n$ | ◯ | ◯ | ◯ |
| $n^3$ | $n^{\log_2 7}$ | ◯ | ◯ | ◯ |
| $2^{\log n}$ | $2^{10 \log n}$ | ◯ | ◯ | ◯ |
| $2^n$ | $2^{2^{\log n}}$ | ◯ | ◯ | ◯ |
| $n!$ | $n^n$ | ◯ | ◯ | ◯ |

2. (**3 points**) Consider the following statement: For $f(n), g(n) > 0$, if $\log f(n) = O(\log g(n))$, then $f(n) = O(g(n))$. Either prove the statement, or give a counterexample with justification.

3. (**3 points**) Give a formal proof that $n = \Omega(\log n)$.

## 2 Recurrences. (13 points.)

Please give the tightest big O bound that you can.

1. $T(n) = 9T(n/3) + O(n^3)$

2. $T(n) = 9T(n/3) + O(n^2)$

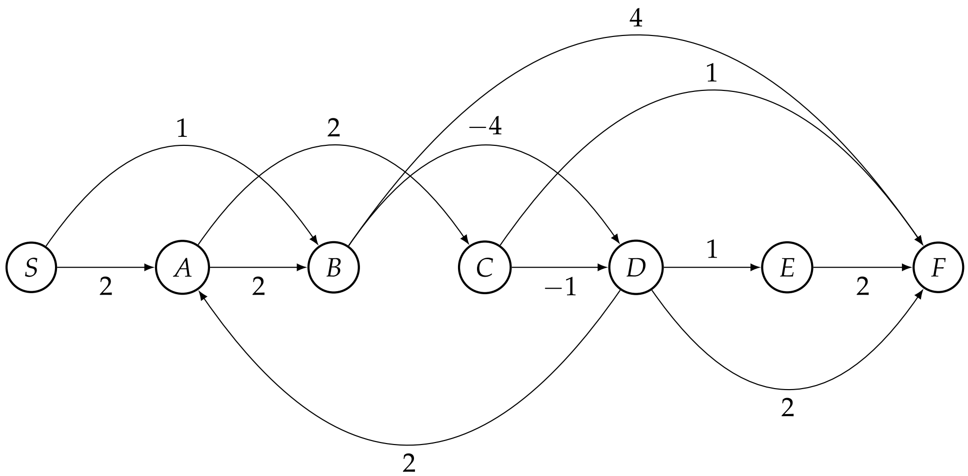
3. $T(n) = 4T(n-2) + O(2^n)$

4. $T(n) = T(n/3) + T(2n/3) + O(n)$

5. $T(n) = 3T(n^{1/3}) + O(\log n)$

## 3 Update, here, there, not quite everywhere. (6 points.)

Consider the graph below with the edge weights as indicated.




1. Give the shortest path distances to each vertex from $S$. (Be sure to notice all edges are directed from left to right, except the arc from $D$ to $A$ which is directed from right to left.)

   S: A: B: C:

   D: E: F:

2. Give a sequence of 6 edge updates that starting from $d(S) = 0$, and $d(X) = \infty$ for all $X \neq S$, produces a valid set of shortest path distances. (We give you the first one.)

   Update 1: $(S, B)$ Update 2:

   Update 3: Update 4:

   Update 5: Update 6:

# 4 Short answers and True/False. (38 points. 2 points each part.)

1. Let $\omega_n$ be the $n$th primitive root of unity in the complex numbers. We let $u^*$ be the vector where each complex number is replaced by its complex conjugate.

   Recall for a complex number $a = re^{i\theta}$, its conjugate is $a^* = re^{-i\theta}$. Notice that $aa^* = r^2 e^0 = r^2$.

   Let $u = (1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1})$ and $v = (1, \omega_n^2, \omega_n^4, \ldots, \omega_n^{2(n-1)})$.

   (a) What is $u \cdot u^*$? (Recall $x \cdot y$ for vectors $x$ and $y$ is $\sum_i x_i y_i$.)

   (b) What is $u \cdot v^*$? (Recall $u^*$ is the vector of conjugates of the elements of $u$)

2. Recall that $\omega_n$ is a primitive $n$th root of unity. That is, $\omega_n = e^{2\pi i/n}$.

   Let $n = 2^k$. What is the smallest $t > 0$ where $(\omega_n^4)^t = 1$?

   (Answer is an expression possibly using $n$ and $k$.)

3. We wish to evaluate $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ for $x \in \{1, 2, 3, 4\}$ as follows. (Fill in the blanks so that the matrix computes the evaluations of $A(x)$)

$$\begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{bmatrix}$$

4. For a DFS postordering for a directed graph, $G = (V, E)$, for every edge $(u, v) \in E$, $post(u) > post(v)$.

   ○ True ○ False

5. For an undirected graph, if there is a vertex of degree at least 3 in a DFS tree, removing that vertex disconnects the graph.

   ○ True ○ False

6. For every graph with a Hamiltonian cycle (a simple cycle which contains every vertex), depth first search could produce a tree of depth $n-1$.

◯ True ◯ False

7. For a DAG with a single source $s$ (a vertex with no incoming edges), all vertices are reachable from $s$.

◯ True ◯ False

8. For any depth first search of a DAG, the vertex with the lowest post-order number is a sink.

◯ True ◯ False

9. If there is a unique minimum weight edge in an undirected connected weighted graph, then it must be in every minimum spanning tree.

◯ True ◯ False

10. If a weighted undirected connected graph remains connected after removing all edges with weight greater than $w$, then every edge in any MST has weight at most $w$.

◯ True ◯ False

11. For every weighted graph $G=(V,E)$ and cut $(S,V-S)$, there is an MST which contains all minimum weight edges across that cut.

◯ True ◯ False

12. Given a graph, $G=(\{1,\ldots,n\},E)$, where $n>3$, and $E$ contains the edges $(1,2)$, $(2,3)$ and $(1,3)$ with edge weights 1, 1, and 2 respectively:

    (a) Which, if any, of the edges $(1,2)$, $(2,3)$, $(1,3)$ must be in every MST? (Give the edge(s) or state None.)

    (b) Which, if any, of the edges $(1,2)$, $(2,3)$, $(1,3)$ cannot be in any MST? (Give the edge(s) or state None.)

13. Recall Bellman-Ford on a graph $G = (V, E)$ from a source $s$ initializes $d(s)$ to 0 and $d(v) = \infty$ for all other vertices $v$. Then, it updates all the edges $|V| - 1$ times.

    (a) If there are at most $k$ negative length edges in $G$, then updating all edges $k$ times is sufficient to compute all shortest path distances from $s$.

    ◯ True ◯ False

    (b) If every shortest path from $s$ to any vertex uses at most $k$ edges,then updating all edges $k$ times is sufficient to compute shortest path distances from $s$.

    ◯ True ◯ False

14. Consider an undirected graph $G$ with vertices $\{A, B, C, D, E, F, H\}$, where a breadth first search computes the following distances:

$$[A : 0], [B : 1], [C : 1], [D : 1], [E : 2], [F : 2], [H : 3].$$

    (a) There cannot be an edge $(B, H)$ in $G$.

    ◯ True ◯ False

    (b) Identify an edge that must exist $G$.

    (c) Identify the smallest set of vertices that is guaranteed to be a vertex cut. (Recall from homework, a vertex cut is a set of vertices whose removal leaves a disconnected graph.)

# 5 Polynomial Multiplication/Applications. (8 points. 2/2/4 points by part.)

1. Given that $A(x) = a_0 + a_1x + a_2x^2$ and $B(x) = b_0 + b_1x + b_2x^2$, what is the coefficient of $x^2$ in $A(x) \times B(x)$? (In terms of $a_0, a_1, a_2, b_0, b_1$ and $b_2$.)

2. Given $A(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$ and $B(x) = b_0 + b_1x + b_2x^2 \ldots a_nx^n$, what is the coefficient of $x^k$ in $A(x) \times B(x)$? (In terms of $a_0, a_1, \ldots a_n$ and $b_0, \ldots, b_n$.)

3. Let $A$ and $B$ be independent random variables which take on values in $[0, \ldots, n-1]$, where $Pr[A = k] = a_k$ and $Pr[B = k] = b_k$.

   Briefly describe an $O(n \log n)$ time algorithm that, given $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$, computes the probability mass function for $A + B$, i.e., computes $Pr[A + B = k]$ for all relevant $k$.

   (Hint: consider all the values of $A$ and $B$ where $A + B = k$.)

## 6 Divide and Conquer. (12 points. 6/6 per part.)

1. Given $k$ sorted lists, each of length $n$, give an algorithm to produce a single sorted list containing the elements of all the lists. The runtime should be $O(kn \log k)$.

**Justify the runtime.**

2. In the MultipleSelect problem that given a set of $n$ items and a set $r$ of select requests $k_1, \ldots, k_r$.

   Consider the algorithm of finding a median element (using deterministic median finding) and recursively computing MultipleSelect on each half with the appropriate requests.

   **Argue that this algorithm takes at most $O(n \log r)$ time.**

   (Note: if there are no requests in one of the halves, one does not need to make a recursive call on that half.)

## 7 Strongly connected components. (6 points.)

Recall, the strongly connected components algorithm on a graph $G$ proceeds by running depth first search on the reverse graph, $G^R$, and then runs depth first search on $G$ using inverse post order number.

Let $G$ have vertices $a, b, c, d, e, f, g, h$. The table below illustrates the post order numbering of the first run, and the post order numbering of the second run on $G$, using the inverse post ordering from the first run.

| | $a$ | $b$ | $c$ | $d$ | e | f | g | h |
|---|---|---|---|---|---|---|---|---|
| $post(G^R)$ | 16 | 8 | 4 | 11 | 9 | 10 | 15 | 14 |
| $post(G)$ | 6 | 13 | 16 | 10 | 14 | 9 | 4 | 5 |

1. What are the strongly connected components (SCCs) of the graph?

2. What is the topological sort of the strongly connected components of the graph?

## 8 Shortest Path: remove an edge. (10 points)

Consider an undirected graph $G = (V, E)$ with non-negative edge weights. You are also given a shortest path labelling, $d(\cdot)$, and a corresponding shortest path tree $\mathcal{T}$.

Consider removing an edge $(u, v)$ in $\mathcal{T}$. Let $S$ be the set of $v$'s descendants in $\mathcal{T}$ and $E(S)$ be the set of edges where at least one of the endpoints is in $S$.

Give an $O(|E(S)| \log |S|)$ algorithm to update the distance labels in $d(\cdot)$ to be the shortest path distances in the resulting graph.

**Justify the runtime.**

## 9 Points on a Line. (6 points.)

You are given a list $L = [x_1, x_2, \ldots, x_n]$ of $n$ points on the real line where $n$ is even and $x_1 < x_2 < \ldots < x_n$. Design a greedy algorithm that partitions $L$ into $n/2$ pairs $(a_i, b_i)$, $i = 1, 2, \ldots, n/2$ to minimize:

$$\sum_{i=1}^{n/2} |a_i - b_i|$$

**Give a description of your algorithm.**

Justify the **correctness** of your algorithm using an **exchange/swapping argument**.