Chemical Engineering 150A
Midterm Exam
Monday, April 8, 2019 6:10 pm – 7:00 pm

The exam is 100 points total.

Name: ________________________________ (in Uppercase)

Student ID: ____________________________

You are allowed one 8.5"×11" sheet of paper with your notes on both sides and a calculator for this exam.

The exam should have 17 pages including the cover page.

Instructions:

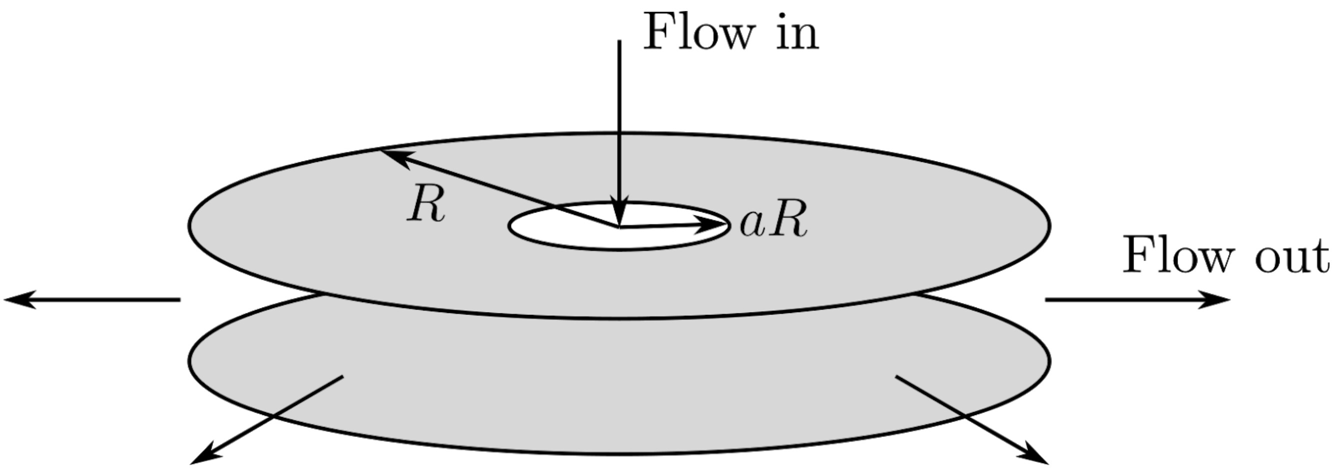
1) Please write your answers in the box if provided.
2) Do your calculations in the space provided for the corresponding part. Any work done outside of specified areas will not be graded.
3) Please sign below saying that you agree to the UC Berkeley honor code.
4) The exam contains two problems with sub-parts.
5) Navier-Stokes, continuity and constitutive equations are provided at the end. If you simplify directly on the handout, we will grade it.
5) Use the blank white full pages behind the question pages as scratch sheets (your work here will not be graded).

Honor Code:
As a member of UC Berkeley, I act with honesty, integrity, and respect for others.

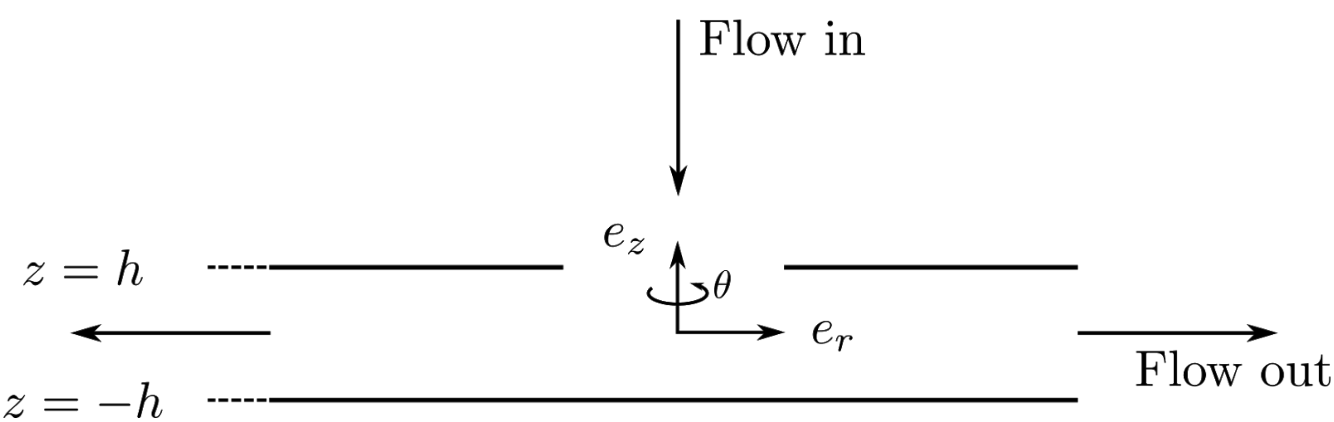
Signature: _

1.a	1.b	1.c	1.d	2.a	2.b	Total

Problem 1. (85 points)
Consider two concentric disks which are fixed to stay in place. There is an inflow of an incompressible fluid through a hole in the top plate. The flow is driven by a pressure drop, $\Delta p = p(r = aR) - p(r = R)$, between the inlet at $r = aR$ ($a \ll 1$) and the outlet at $r = R$. Below, you will find the 3D view of the apparatus.



The following figure shows the front view of the apparatus above. Consider the cylindrical coordinate system to be positioned in the center.



<u>You can make the following assumptions:</u>

1. The outflow is homogeneous along the circumference.
2. The flow is fully developed and at steady-state.
3. The fluid is incompressible.
4. The outlet is at atmospheric pressure p_{atm}.
5. The effects due to flow around the corner at the inlet can be neglected ($a \ll 1$). The pressure at that point is unperturbed by the flow around the corner.

Further note the following:

1. Make all further assumptions that seem physical to you for velocity and pressure dependencies to reduce the complexity. Explicitly state your assumptions motivated from physical arguments in a couple of words/sentences.
2. Use the coordinate system defined in the figure.
3. If you are running out of time, make sure to write the relevant equations and appropriate boundary conditions to get partial credit.

In the following, we will determine the velocity profile in the apparatus.

(a) (10 points)
Write down the kinematics for this problem by stating physically reasonable assumptions.

(i) Which velocity components are non-zero?
(ii) Which coordinates (i.e. r, θ, z) do the non-zero velocity component(s) depend on?

$$\underline{v} = v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z$$

(i)

$$v_r \neq 0$$
$$v_\theta = 0$$
$$v_z = 0$$

(ii)

$$v_r = v_r (r, z)$$

(b) (20 points)
Use the local form of the balance of mass to determine the radial component of the velocity profile in terms of an integration constant. Write your final answer in the box provided below.

The microscopic form of the balance of mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

Using the steady-state assumption and rewriting in cylindrical coordinates . yields

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Using the kinematic assumption from (a) and $\rho = \text{constant}$ gives the differential equation

$$\begin{aligned} &\frac{\partial}{\partial r}(r v_r) = 0 \\ \Leftrightarrow \quad & r v_r = C(z) \\ \Leftrightarrow \quad & v_r = \frac{C(z)}{r} \end{aligned}$$

$C(z)$ is an integration constant that depends on z, i.e. it is only constant with respect to the integration variable r.

(c) (50 points)
Use the Navier-Stokes equations to find the velocity profile. Start by simplifying them by using your results from (a) and (b). When solving the resulting differential equation, state and apply appropriate boundary conditions.
In order to be able to find an analytical solution, neglect the nonlinear term ($\rho C^2/r^3$, where C is your integration constant from (b)). Write your final expression for the velocity in the box provided below.

Note: Express your final answer in terms of $\Delta p,\ \mu,\ a,\ R,\ h,\ z,\ r$.

Hint 1: In multivariable calculus, an "integration constant" is only a constant with respect to the integration variable. Consider this fact when you use your result from part (b).

Hint 2: After neglecting the nonlinear term, the differential equation that will determine the velocity profile should be of the following mathematical form:

$$0 = \frac{\partial f(r,z)}{\partial r} + K\frac{1}{r}\frac{\partial^2 g(z)}{\partial z^2} ,$$

where $f(r,z)$ is some function of r and z, $g(z)$ is some function of z, and K is some constant.

Hint 3: Note that we are not asking you to find the pressure profile.

We start with the vectorial form of the Navier-Stokes equations.

$\rho\left(\frac{\underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$

We only need to investigate the radial direction of the Navier Stokes equation,

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right)=$$
$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)+\rho g_r$$

Using the steady-state assumptions, the functional dependence and kinematic assumptions from (a) give

$$\frac{\partial v_r}{\partial t}=0$$
$$\frac{\partial v_r}{\partial \theta}=0$$
$$v_\theta=0$$
$$v_z=0$$
$$g_r=0.$$

Using these relations simplifies the radial direction of the Navier-Stokes equations to

$$\rho v_r\frac{\partial v_r}{\partial r}=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial^2 v_r}{\partial z^2}\right)$$

Using the result from (b), we find

$$\frac{\partial v_r}{\partial r}=\frac{\partial}{\partial r}\frac{C(z)}{r}=-\frac{C(z)}{r^2}$$

and

$$\frac{\partial}{\partial r}(rv_r)=\frac{\partial C(z)}{\partial r}=0,$$

such that we can simplify the r-direction of the Navier-Stokes equation further to

$$-\rho\frac{C(z)^2}{r^3}=-\frac{\partial p}{\partial r}+\frac{\mu}{r}\frac{\partial^2 C(z)}{\partial z^2}.$$

Ignoring the nonlinear term gives

$$0=-\frac{\partial p}{\partial r}+\frac{\mu}{r}\frac{\partial^2 C(z)}{\partial z^2}.$$

This equation can now by integrated in the radial direction, yielding

$$0=\int_{aR}^{R}-\frac{\partial p}{\partial r}+\frac{\mu}{r}\frac{\partial^2 C(z)}{\partial z^2}\mathrm{d}r=-\Delta p+\mu\ln\left(\frac{R}{aR}\right)\frac{\partial^2 C(z)}{\partial z^2}$$

Integrating twice in the z-direction then results in

$$0 = -\Delta p \frac{z^2}{2} + \mu C(z) \ln\left(\frac{1}{a}\right) - C_2(r) z - C_3(r),$$

where C_2 and C_3 are integration constants, which could depend on r. However, solving for $C(z)$ yields

$$C(z) = \left(\Delta p \frac{z^2}{2} + C_2(r) z + C_3(r)\right) \frac{1}{\mu \ln\left(\frac{1}{a}\right)}$$

and noting that this expressions holds for all values $-h \leq z \leq h$ shows that C_2 and C_3 are in fact independent of r. We then find

$$v_r(r, z) = \left(\Delta p \frac{z^2}{2} + C_2 z + C_3\right) \frac{1}{r \mu \ln\left(\frac{1}{a}\right)}.$$

Next, we write out the boundary conditions required to determine the integration constants. The no-slip boundary conditions on the top and bottom plate are

$$v_r|_{z=h} = 0$$
$$v_r|_{z=-h} = 0.$$

This yields the two equations

$$0 = \left(\Delta p \frac{h^2}{2} + C_2 h + C_3\right)$$
$$0 = \left(\Delta p \frac{h^2}{2} - C_2 h + C_3\right)$$

Adding gives

$$C_3 = -\frac{\Delta p h^2}{2}$$

and subtracting yields

$$C_2 = 0.$$

The latter reflects the symmetry of the problem over $z = 0$. Using

$$\ln\left(\frac{1}{a}\right) = -\ln(a),$$

we find

$$v_r(r,z) = \frac{\Delta p h^2}{2\mu r \ln(a)}\left(1 - \left(\frac{z}{h}\right)^2\right)$$

(d) **(5 points)**
In (c), we asked you to neglect the nonlinear term. In discussion section 10, we derived the non-dimensional form of the Navier-Stokes equations to be

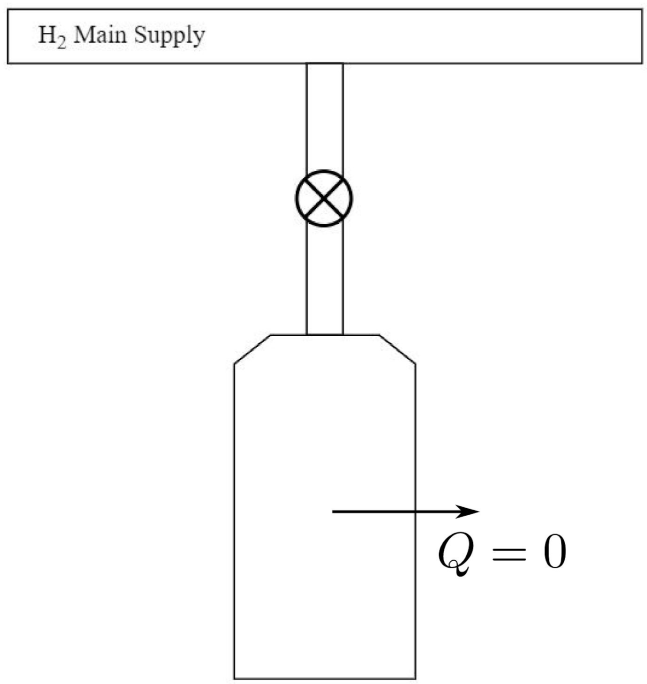
$$Re\frac{\partial \underline{v}^\star}{\partial t^\star} + Re\underline{\nabla}^\star \cdot (\underline{v}^\star \otimes \underline{v}^\star) = -\underline{\nabla}^\star p^{\star\star} + (\underline{\nabla}^\star)^2 \underline{v}^\star + \frac{Re}{Fr}\underline{g}^\star ,$$

where Re and Fr are the Reynolds and Froude Number respectively. The asterisks indicate non-dimensional quantities. Using the above non-dimensional form of the Navier-Stokes equations, explain in words the meaning of neglecting the nonlinear term in part (c).

In the discussion section, it has been shown the equation is non-dimensionalised such that all terms are of $\mathcal{O}(1)$, except for the non-dimensional numbers. In this problem, gravity does not act in the radial direction and is thus zero. The left-hand side is multiplied by the Reynolds' number, whereas the right-hand side is $\mathcal{O}(1)$. The nonlinear term that we ignored stems from the second term on the left-hand side, thus implying that $Re \ll 1$. This implies that the viscous forces dominate over the inertial forces. This is called Creeping Flow and the corresponding equation is called Stokes equation.

Problem 2. (15 points)

Suppose there is a new company responsible for supplying H_2 gas cylinders to customers in the Bay Area. They guarantee a minimum pressure of 2100 psig in all of their cylinder products. They produce these products by adiabatically filling an initially empty cylinder with H_2 from a main supply line through a valve. When the pressure inside the cylinder reaches the target pressure (2100 psig), the valve is closed and flow is ended. The gas in the main supply line as well as the outside temperature are at room temperature. However, their customer has complained that the pressure in the cylinder they received was only 1600 psig upon arrival to their site.



(a) (10 points)
Explain, in six sentences or less, why the pressure was lower than expected when the cylinder reached the site, which is also at room temperature. Assume H_2 can be treated as an ideal gas.

During the filling process, the incoming gas is performing work on the gas in the system, and thus the cylinder heats up. Since the process is adiabatic, the heat continues building up in the cylinder leading to a final temperature that is much higher than the ambient temperature once filling is complete. The final pressure in the cylinder is 2100 psig at this elevated temperature, but once the cylinder arrives at the customer's site and has equilibrated back down to ambient temperature, the pressure will have decreased (based on ideal gas law: PV = nRT).

(b) (5 points)
Based on your answer to (a), indicate how the cylinder filling process should be changed in order to avoid this issue. Answer in four sentences or less.

The cylinder filling process should be changed to occur under isothermal conditions, instead of adiabatic conditions.

The cylinder could be filled up to a higher pressure such that it would reach the required pressure once cooled down to room temperature.

SCRATCH SHEET

SCRATCH SHEET

SCRATCH SHEET