Second Midterm Examination
Tuesday July 23 2019
Closed Books and Closed Notes

Question 1

Planar Oscillations of a System of Two Particles *(25 Points)*

Consider a particle of mass m_2 which is suspended by a rod of length ℓ from a point A by a pin joint (see Figure 1). The pin joint is rigidly mounted to a box of mass m_1. The box, which is free to move on a horizontal track, is attached to a fixed point O by a linear spring of stiffness K and unstretched length ℓ_0 and is under the influence of an applied force

$$\mathbf{P} = P(t)\mathbf{E}_x \text{ where } P(t) = P_0 \cos(\omega t). \tag{1}$$

Both the box and the particle are under the influence of the respective gravitational forces $-m_1 g\mathbf{E}_y$ and $-m_2 g\mathbf{E}_y$. The box is modeled as a particle of mass m_1 that is constrained to undergo purely translational motions in the $\mathbf{E}_x$ direction.

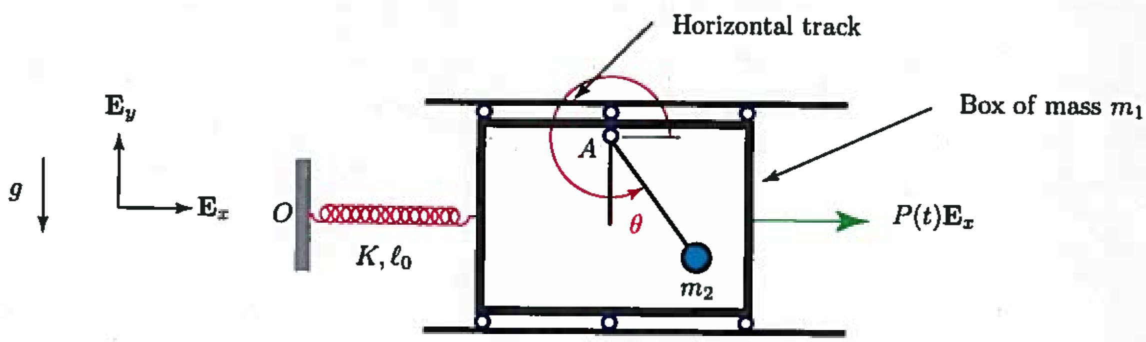


Figure 1: *A pendulum of length ℓ and mass m_2 that is suspended from a box of mass m_1. The system is being driven into motion by an applied force* $\mathbf{P}(t)$.

Using a cylindrical polar coordinate system, the positions of the particles can be expressed as

$$\mathbf{r}_1 = \mathbf{r}_A = (x + c_1)\,\mathbf{E}_x, \qquad \mathbf{r}_2 = \mathbf{r}_A + \ell\mathbf{e}_r. \tag{2}$$

The constant c_1 in the expression for $\mathbf{r}_1$ is chosen so that the extension ϵ of the spring is given simply by

$$\epsilon = x. \tag{3}$$

(a) *(4 +6 Points)* Show that the linear momentum and kinetic energy of the system have the representations

$$\mathbf{G} = (m_1 + m_2)\,\dot{x}\mathbf{E}_x + m_2\ell\dot{\theta}\mathbf{e}_\theta, \qquad T = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}???\dot{\theta}^2 - m_2\ell\dot{\theta}\dot{x}??. \tag{4}$$

For full credit supply the missing terms.

(b) *(4 + 4 Points)* Draw a pair of freebody diagrams: one for the system of particles and one for the particle of mass m_2. For full credit provide a clear expression for the spring force.

(c) *(7 Points)* Using the work-energy theorem $\dot{T} = \mathbf{F}_1 \cdot \mathbf{v}_1 + \mathbf{F}_2 \cdot \mathbf{v}_2$ show that the total energy E of the system is not conserved. For full credit, provide an expression for E.

Question 2

Dynamics of a Dumbbell (30 Points)

As shown in Figure 2, a particle of mass m_1 and a second particle of mass m_2 are connected together by a massless rod of length ℓ. A constant force $P_0\mathbf{E}_y$ acts on one of the particles as shown in the figure. To describe the kinematics of this system, the position vector of the center of mass C is described using a set of Cartesian coordinates and the position vectors of m_1 and m_2 relative to C are described using a set of cylindrical polar coordinates:

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y, \qquad \mathbf{r}_1 - \mathbf{r} = \frac{m_2\ell}{m_1+m_2}\mathbf{e}_r, \qquad \mathbf{r}_2 - \mathbf{r} = -\frac{m_1\ell}{m_1+m_2}\mathbf{e}_r. \tag{5}$$

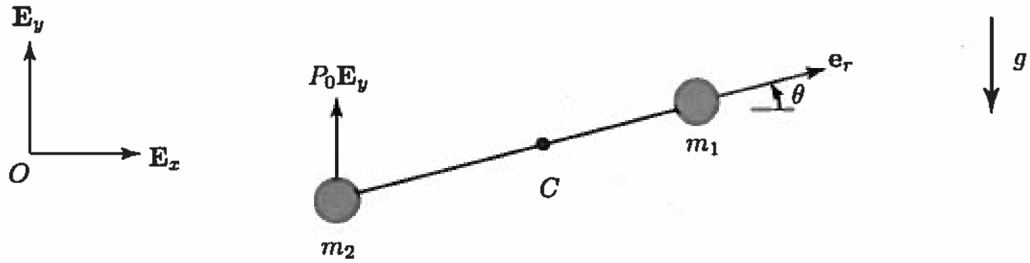


Figure 2: A system of two particles connected by a massless rod of length ℓ. The particles are free to move on a smooth vertical plane.

(a) *(2 + 5 +3+5 Points)* Starting from the representations (5) and using the definitions of the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}$ relative to the center of mass, angular momentum $\mathbf{H}_O$ relative to the fixed point O, and kinetic energy T, show that

$$\mathbf{G} = (m_1+m_2)\left(\dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y\right), \qquad \mathbf{H} = \frac{m_1 m_2}{m_1+m_2}??\mathbf{E}_z, \qquad \mathbf{H}_O = ???\mathbf{E}_z,$$
$$T = \frac{m_1+m_2}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(????). \tag{6}$$

For full credit, supply the missing terms.

(b) *(5 Points)* Draw two freebody diagrams: one for each of the two particles. For full credit, provide clear expressions for the tension force on the particles.

(c) *(6 + 4 Points)*

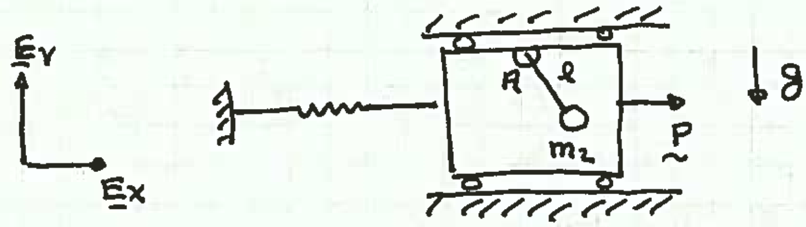
(i) Using an angular momentum theorem, show that

$$\frac{m_1 m_2}{m_1+m_2}\left(???\ddot{\theta}\right) = -\frac{P_0 m_1 \ell}{m_1+m_2}????. \tag{7}$$

For full credit, supply the pair of missing terms.

(ii) Using the work-energy theorem $\dot{T} = \mathbf{F}_1\cdot\mathbf{v}_1 + \mathbf{F}_2\cdot\mathbf{v}_2$, prove that the total energy of the system is conserved. For full credit, supply an expression for E.

Problem 7



$\underline{r}_1 = \underline{r}_A = (x + c_1)\,\underline{E}_x \qquad \underline{r}_2 = \underline{r}_1 + \ell\,\underline{e}_r$

(a) $\underline{G} = m_1 \underline{v}_1 + m_2 \underline{v}_2 = m_1 \dot{x}\,\underline{E}_x + m_2(\dot{x}\,\underline{E}_x + \ell\dot{\theta}\,\underline{e}_\theta)$

$= (m_1 + m_2)\dot{x}\,\underline{E}_x + m_2 \ell \dot{\theta}\,\underline{e}_\theta$

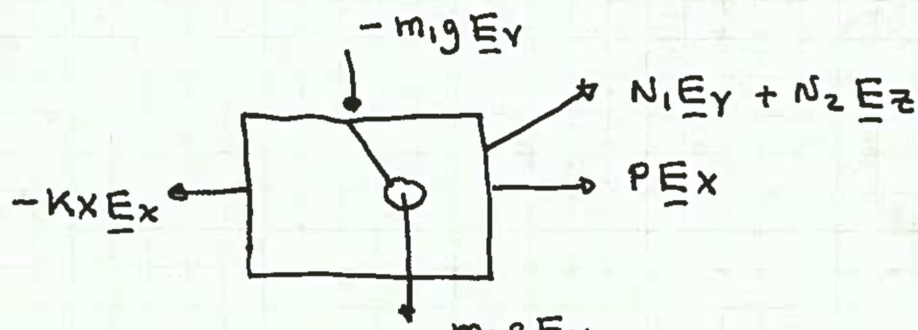
$T = \frac{1}{2} m_1 \underline{v}_1 \cdot \underline{v}_1 + \frac{1}{2} m_2 \underline{v}_2 \cdot \underline{v}_2$

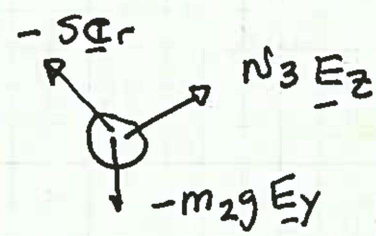
$= \frac{(m_1 + m_2)}{2}\dot{x}^2 + \frac{m_2}{2}\ell^2\dot{\theta}^2$

$- \; m_2 \dot{x} \ell \dot{\theta} \sin\theta$

$\underline{v}_1 = \dot{x}\,\underline{E}_x$
$\underline{v}_2 = \dot{x}\,\underline{E}_x + \ell\dot{\theta}\,\underline{e}_\theta$
$\underline{e}_\theta \cdot \underline{E}_x = -\sin\theta$

(b)





(c)

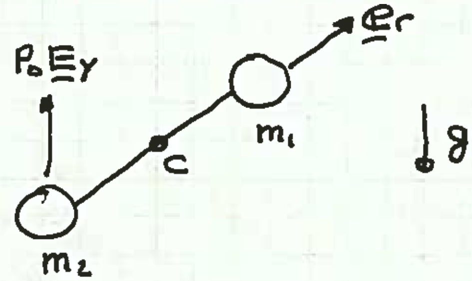
$$
\begin{aligned}
\dot{T} = \; & -Kx\,\underline{E}_x\cdot\underline{v}_1 - m_1 g\,\underline{E}_y\cdot\underline{v}_1 + (N_1\underline{E}_y + N_2\underline{E}_z)\cdot\underline{v}_1 \\
& + S\underline{e}_r\cdot\underline{v}_1 + P\underline{E}_x\cdot\underline{v}_1 - S\underline{e}_r\cdot\underline{v}_2 \\
& - m_2 g\,\underline{E}_y\cdot\underline{v}_2 + N_3\underline{E}_z\cdot\underline{v}_2 \\
= \; & -Kx\dot{x} - 0 + 0 + S\underline{e}_r\cdot(\underline{v}_1 - \underline{v}_2 = -\ell\dot{\theta}\,\underline{e}_\theta) \\
& + P\dot{x} - m_2 g \ell \dot{\theta}\cos\theta + 0 \\
= \; & -\frac{d}{dt}\left(\frac{K}{2}x^2 + m_2 g \ell \sin\theta\right) + P\dot{x}
\end{aligned}
$$

Hence

$$\frac{d}{dt}\left(T + U = E\right) = P\dot{x} \neq 0$$

$\Rightarrow$ $E = T + \frac{1}{2}Kx^2 + m_2 g \ell \sin\theta$ is not conserved.

$\boxed{\text{Problem 2}}$



$$\underline{r} = X\underline{E}_X + y\underline{E}_Y$$

$$\underline{r}_1 = \underline{r} + \frac{m_2}{m_1+m_2}\,\ell\underline{e}_r$$

$$\underline{r}_2 = \underline{r} - \frac{m_1}{m_1+m_2}\,\ell\underline{e}_r$$

(a) $$\underline{G} = m_1\dot{\underline{r}}_1 + m_2\dot{\underline{r}}_2 = (m_1+m_2)\dot{\underline{r}} = (m_1+m_2)(\dot{X}\underline{E}_X + \dot{y}\underline{E}_Y)$$

$$\underline{H}_c = (\underline{r}_1 - \underline{r}) \times m_1(\underline{v}_1 - \underline{v}) + (\underline{r}_2 - \underline{r}) \times m_2(\underline{v}_2 - \underline{v})$$

$$= \frac{m_2\ell}{m_1+m_2}\,\ell\underline{e}_r \times \frac{m_1 m_2}{m_1+m_2}\,\ell\dot{\theta}\underline{e}_\theta$$

$$+ \frac{m_2\ell}{m_1+m_2}\,\ell\underline{e}_r \times \frac{m_1 m_2}{m_1+m_2}\,(-\ell\dot{\theta}\underline{e}_\theta)$$

$$= \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2}\,\ell^2\dot{\theta}\,\underline{E}_z$$

$$= \frac{m_1 m_2}{m_1+m_2}\,\ell^2\dot{\theta}\,\underline{E}_z$$

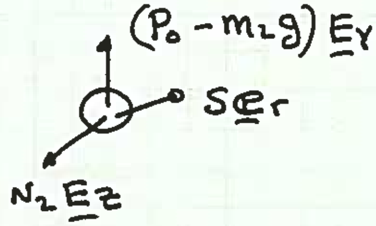
$$\underline{H}_0 = \underline{H}_c + \underline{r}\times m\underline{v} = \underline{H}_c + (m_1+m_2)(X\dot{y} - y\dot{X})\underline{E}_z$$

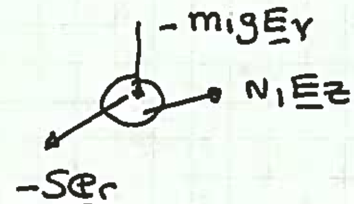
$$T = \tfrac{1}{2}(m_1+m_2)\,\underline{v}\cdot\underline{v} + \tfrac{1}{2}m_1(\underline{v}_1-\underline{v})\cdot(\underline{v}_1-\underline{v}) + \tfrac{1}{2}m_2(\underline{v}_2-\underline{v})\cdot(\underline{v}_2-\underline{v})$$

$$= \tfrac{1}{2}(m_1+m_2)(\dot{X}^2+\dot{y}^2) + \tfrac{1}{2}m_1\left(\frac{m_2^2\ell^2\dot{\theta}^2}{(m_1+m_2)^2}\right) + \tfrac{1}{2}m_2\left(\frac{m_1^2\ell^2\dot{\theta}^2}{(m_1+m_2)^2}\right)$$

$$= \tfrac{1}{2}(m_1+m_2)(\dot{X}^2+\dot{y}^2) + \frac{1}{2}\,\frac{m_1 m_2}{m_1+m_2}\,\ell^2\dot{\theta}^2$$

(b)





(c) (i) $\underline{M} = \dot{\underline{H}}$

$$\begin{aligned}\underline{M} &= (\underline{r}_1 - \underline{r})\times\underline{F}_1 + (\underline{r}_2 - \underline{r})\times\underline{F}_2 \\ &= \frac{m_2 \ell}{m_1+m_2}\underline{e}_r \times\left(-m_1 g\underline{E}_Y - S\underline{e}_r + N_1\underline{E}_Z\right) \\ &\quad - \frac{m_1 \ell}{m_1+m_2}\underline{e}_r \times\left((P_0 - m_2 g)\underline{E}_Y + S\underline{e}_r + N_2\underline{E}_Z\right)\end{aligned}$$

Hence

$$\underline{M} = -\frac{m_1 \ell}{m_1+m_2}\underline{e}_r \times P_0\underline{E}_Y + \frac{\ell\,\underline{e}_r}{m_1+m_2}\times\left(m_2 N_1\underline{E}_Z - m_1 N_2\underline{E}_Z\right)$$

$$\underline{M}\cdot\underline{E}_Z = -\frac{m_1\ell}{m_1+m_2}P_0\cos\theta \qquad \underline{e}_r = \cos\theta\underline{E}_X + \sin\theta\underline{E}_Y$$

From $\underline{M}\cdot\underline{E}_Z = \dot{\underline{H}}\cdot\underline{E}_Z$

$$\frac{m_1 m_2}{m_1+m_2}\ell^2\ddot{\theta} = -\frac{m_1\ell}{m_1+m_2}P_0\cos\theta$$

(ii)

$$\begin{aligned}\dot{T} &= \underline{F}_1\cdot\underline{v}_1 + \underline{F}_2\cdot\underline{v}_2 \\ &= N_1\underline{E}_Z\cdot\underline{v}_1 + N_2\underline{E}_Z\cdot\underline{v}_2 + S\underline{e}_r\cdot(\underline{v}_2 - \underline{v}_1) \\ &\quad - m_1 g\underline{E}_Y\cdot\underline{v}_1 - m_2 g\underline{E}_Y\cdot\underline{v}_2 + P_0\underline{E}_Y\cdot\underline{v}_2 \\ &= 0 + 0 + 0 \\ &\quad - \frac{d}{dt}\left((m_1+m_2)g\underline{E}_Y\cdot\underline{r} - P_0\underline{E}_Y\cdot\underline{r}_2\right)\end{aligned}$$

$\Rightarrow$ $E = T + U$ is conserved where

$$E = T + (m_1+m_2)g\underline{E}_Y\cdot\underline{r} - P_0\underline{E}_Y\cdot\underline{r}_2$$