EE 120: SIGNALS AND SYSTEMS, FALL 2019

Midterm 1, October 3, Thursday, 12:10-2:00 pm

Name: Solutions

SID #: ____________________

Important Instructions:

Closed book. As announced before, you can use one letter-size, two-sided cheatsheet.

Show all your work. An answer without explanation is not acceptable and does not guarantee any credit.

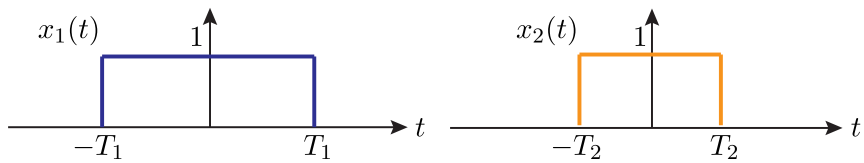
You can use the back of the pages if you need more space.

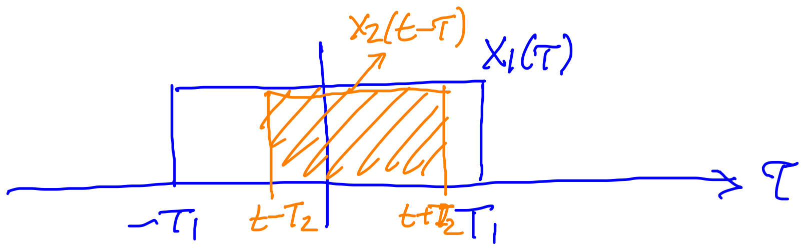
Do not remove pages, as this disrupts the scanning. Instead, cross the parts that you don't want us to grade.

Problem	Points
1	20
2	20
3	20
4	20
5	20
Total	100

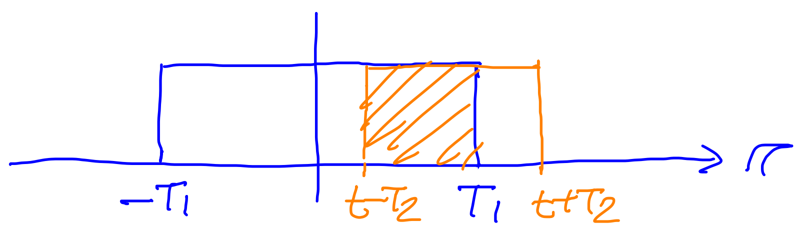
1. (Convolution)

a) (10 points) Find the convolution $x_1 * x_2$ for the signals x_1 and x_2 below, where $T_1 > T_2$. Plot and clearly label $(x_1 * x_2)(t)$.

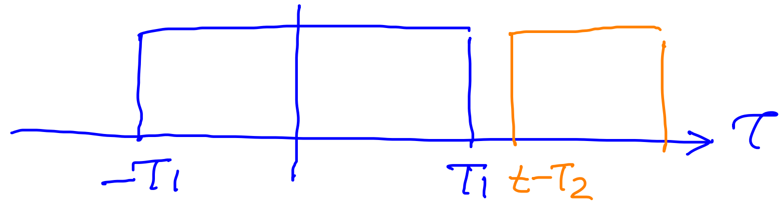




Area $= 2T_2$ if $|t| \leq T_1 - T_2$

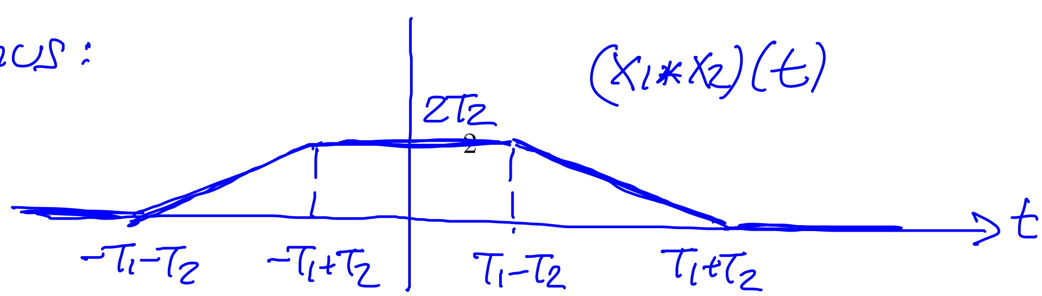


Area $= T_1 + T_2 - |t|$ if $T_1 \leq |t| \leq T_1 + T_2$



Area $= 0$ (no overlap) if $|t| \geq T_1 + T_2$

Thus:



b) (5 points) Express the integral $\int_{-\infty}^{t-1} x(\tau)d\tau$ as a convolution $(x * g)(t)$ with a signal g that you will determine.

$$\int_{-\infty}^{t-1} x(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\, k(t,\tau)\, d\tau$$

where $k(t,\tau) = \begin{cases} 1 & \tau \le t-1 \\ 0 & \tau > t-1. \end{cases}$

Note $k(t,\tau) = \begin{cases} 1 & t-\tau \ge 1 \\ 0 & t-\tau < 1 \end{cases}$

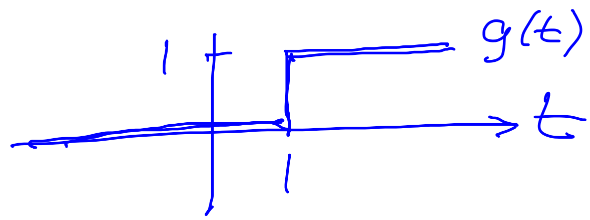
$$= u(t-\tau-1)$$

where $u(\cdot)$ is the unit step.

Therefore we can write the integral as

$$(x * g)(t)$$

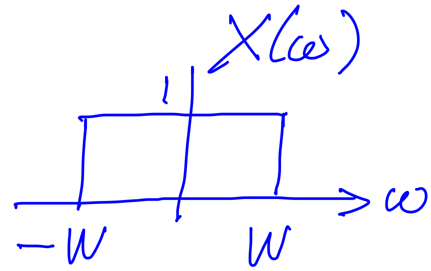
where $g(t) = u(t-1)$.



c) (5 points) Find a continuous-time signal x, not identically equal to zero, such that $x * x = x$.

Recall

$$x(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}t\right) \overset{FT}{\longleftrightarrow}$$



From the convolution property of FT,

$$(x * x)(t) \longleftrightarrow X(\omega)X(\omega) = X(\omega).$$

Therefore $x(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}t\right)$ is a valid answer for any W, including the unit impulse δ which we recover as $W \to \infty$.

2. (LTI Systems) Consider a LTI system defined by the difference equation:

$$y[n] - 1.1y[n-1] + 0.3y[n-2] = 2x[n] - x[n-1]. \qquad (1)$$

a) (6 points) Find the frequency response $H(e^{j\omega})$ for (1).
b) (8 points) Find the impulse response $h[n]$ for (1).
c) (6 points) Determine if (1) is a stable system.

a) Since $x[n] = e^{j\omega n}$ gives

$$y[n] = H(e^{j\omega})e^{j\omega n},$$

we can substitute these in (1):

$$\left(1 - 1.1e^{-j\omega} + 0.3e^{-j2\omega}\right) H(e^{j\omega})\cancel{e^{j\omega n}} = \left(2 - e^{-j\omega}\right)\cancel{e^{j\omega n}}$$

and obtain:

$$H(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 - 1.1e^{-j\omega} + 0.3e^{-2j\omega}}$$

b) $H(e^{j\omega})$ above is a valid answer but it actually simplifies more:

$$= \frac{2\cancel{(1 - 0.5e^{-j\omega})}}{\cancel{(1 - 0.5e^{-j\omega})}(1 - 0.6e^{-j\omega})} = \frac{2}{1 - 0.6e^{-j\omega}}$$

Therefore, $h[n] = 2(0.6)^n u[n]$

Additional space for Problem 2.

c) Stable because

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 2(0{,}6)^n < \infty .$$

$$= 2 \frac{1}{1-0{,}6} = \frac{2}{0{,}4} = 5$$

3. (Fourier Series) Given the period-10 sequence $x[n]$ depicted below, determine the following quantities where a_k denotes the kth Fourier series coefficient:

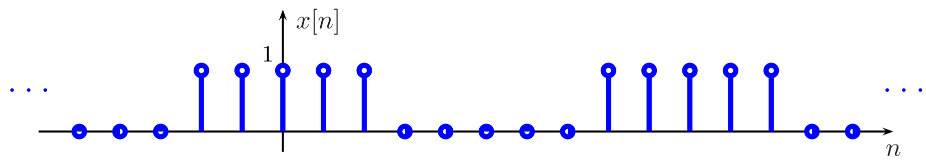
a) (4 points) a_0,

b) (4 points) a_5,

c) (4 points) Imaginary part of a_7.

d) (4 points) $\sum_{k=0}^{9} a_k$,

e) (4 points) $\sum_{k=0}^{9}(-1)^k a_k$.



a) $a_0 = \frac{1}{N}\sum_{\langle N\rangle} x[n] = \frac{5}{10} = 0.5$

b) $a_5 = \frac{1}{10}\sum_{\langle 10\rangle} x[n] e^{-j\frac{2\pi}{10}5n} = \frac{1}{10}\sum_{\langle 10\rangle} x[n](-1)^n$

$= \frac{1}{10}(1-1+1-1+1) = 0.1$

c) x is even symmetric

so a_k is real for all $k \Rightarrow Im\{a_k\} = 0$.

d) $\sum_{k=0}^{9} a_k = \sum_{k=0}^{9} a_k e^{j\frac{2\pi}{N}kn}\Big|_{n=0} = x[n]\Big|_{n=0} = x[0] = 1$

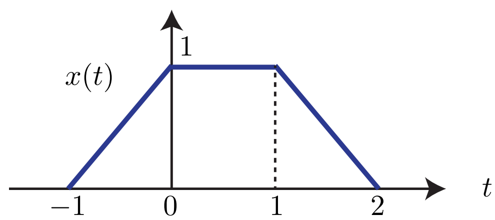
e) $\sum_{k=0}^{9}(-1)^k a_k = \sum_{k=0}^{9} a_k e^{j\pi k} = \sum_{k=0}^{9} a_k e^{j\frac{2\pi}{10}kn}\Big|_{n=5} = x[5]$

$= 0$

Additional space for Problem 3.

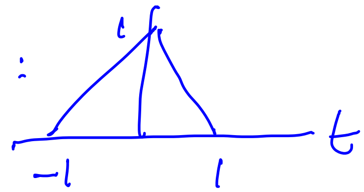
4. (Properties of the Fourier Transform) To answer the following questions you can use relevant properties of the Fourier Transform along with Fourier Transform pairs already derived in class.

a) (5 points) Find the Fourier Transform of the signal shown below.



Many ways to do this. For examples, we showed in class:

$g(t)$:



$\longleftrightarrow G(\omega) = \text{sinc}^2\left(\frac{\omega}{2\pi}\right)$

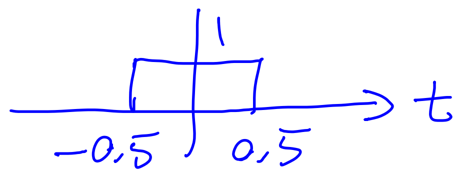
Note that $x(t) = g(t) + g(t-1)$ and $g(t-1) \longleftrightarrow e^{-j\omega} G(\omega)$. By linearity,

$$X(\omega) = \left(1 + e^{-j\omega}\right) G(\omega)$$
$$= \left(1 + e^{-j\omega}\right) \text{sinc}^2\left(\frac{\omega}{2\pi}\right)$$

b) (5 points) Find the Fourier Transform of the signal:

$$x(t) = \begin{cases} \cos(10\pi t) & |t| \le 0.5 \\ 0 & |t| > 0.5. \end{cases}$$

$x(t) = \cos(10\pi t) \cdot$



$\Updownarrow$ FT $\qquad \Updownarrow$ FT

Multiplication Property:

$$X(\omega) = \frac{1}{2\pi}\left(\pi\,\delta(\omega-10\pi)+\pi\,\delta(\omega+10\pi)\right) * \left(\text{sinc}\left(\frac{\omega}{2\pi}\right)\right)$$

$$= \frac{1}{2}\,\delta(\omega-10\pi) * \text{sinc}\left(\frac{\omega}{2\pi}\right) + \frac{1}{2}\,\delta(\omega+10\pi) * \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$= \frac{1}{2}\,\text{sinc}\left(\frac{\omega-10\pi}{2\pi}\right) + \frac{1}{2}\,\text{sinc}\left(\frac{\omega+10\pi}{2\pi}\right)$$

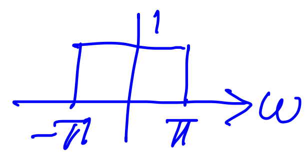
c) (5 points) Evaluate the integral:

$$\int_{-\infty}^{\infty} \text{sinc}^2(t)dt.$$

Parseval's Relation:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Since $\text{sinc}\, t \Longleftrightarrow$



$$\int_{-\infty}^{\infty} \text{sinc}^2(t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = \frac{1}{2\pi} 2\pi = 1$$

d) (5 points) Prove the following property of convolution:

$$(f * g)' = f' * g = f * g'$$

where $'$ represents the derivative: $f'(t) := df(t)/dt$.

Note $f'(t) \overset{FT}{\longleftrightarrow} j\omega F(\omega)$ — (1)

and $g'(t) \longleftrightarrow j\omega G(\omega)$ — (2)

from the derivative property.

Also $(f * g)(t) \longleftrightarrow F(\omega)G(\omega)$

from the convolution property.

Thus, $(f*g)'(t) \longleftrightarrow j\omega F(\omega)G(\omega)$.

If we view this as $(j\omega F(\omega))G(\omega)$

then the inverse FT is $f' * g$

from (1) and the convolution property.

If we view it as $F(\omega)(j\omega G(\omega))$ then

the inverse FT is $f * g'$. Thus

$$(f*g)' = f' * g = f * g'.$$

5. Define the cosine transform by:

$$X^{\cos}(\omega) := \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt.$$

a) (7 points) Show that, when x is real valued, the cosine transform is the real part of the Fourier Transform: $X^{\cos}(\omega) = \mathrm{Re}\{X(\omega)\}$.

b) (6 points) Show that the cosine transform is linear. In other words, for any two signals x_1, x_2, and constants α, β, the cosine transform of $\alpha x_1 + \beta x_2$ is $\alpha X_1^{\cos} + \beta X_2^{\cos}$, where $X_i^{\cos}$ is the cosine transform of x_i, $i = 1, 2$.

c) (7 points) Show that $X^{\cos}(\omega) = 0$ for all ω when x is real and odd-symmetric: $x(-t) = -x(t)$ for all t.

a) Substitute $\cos\omega t = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$.

Then $X^{\cos}(\omega) = \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt}_{\substack{X(-\omega) \\ = X^*(\omega) \text{ since } x \text{ real} \\ \text{(conjugate symmetry property)}}} + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}_{X(\omega)}$.

Thus, $X^{\cos}(\omega) = \frac{1}{2}(X^*(\omega) + X(\omega))$
$= \mathrm{Re}\{X(\omega)\}$.

b) $\int_{-\infty}^{\infty} (\alpha x_1(t) + \beta x_2(t))\cos\omega t\, dt$
$= \alpha \int_{-\infty}^{\infty} x_1(t)\cos\omega t\, dt + \beta \int_{-\infty}^{\infty} x_2(t)\cos\omega t\, dt$
$= \alpha X_1^{\cos}(\omega) + \beta X_2^{\cos}(\omega)$.

c) When x is real and odd symmetric (i.e, $x(-t) = -x(t)$) $Re\{X(\omega)\} = 0$.

Therefore $X^{cos}(\omega) = Re\{X(\omega)\} = 0$,

Alternatively you can directly observe the odd symmetry in the integral:

$$X^{cos}(\omega) = \int_{-\infty}^{\infty} x(t) \cos \omega t \, dt.$$

Since $x(-t)\cos(-\omega t) = x(-t)\cos \omega t$
$= -x(t) \cos \omega t$

the integral $\int_{-\infty}^{0}$ cancels out $\int_{0}^{\infty}$.