

1. (100) In a certain plane flow, the fluid velocity  $\mathbf{V} = v_x\mathbf{i} + v_y\mathbf{j}$  is given by

$$v_x = kyx^2, \quad v_y = -kxy^2, \quad (1.1)$$

where  $k > 0$  is constant.

- (a) Show that (1.1) satisfies the no-slip condition on  $x = 0$ , and also on  $y = 0$ .  
 (b) Find, and sketch, the streamlines.  
 (c) Calculate the components  $a_x$  and  $a_y$  of the fluid acceleration. On your sketch in part (b), show the position vector  $\mathbf{r}$  and the fluid acceleration  $\mathbf{a}$ .  
 (d) In an arbitrary flow of a Newtonian fluid, the shear stress  $\tau$  exerted by the fluid in the  $x$ -direction on a surface whose normal is in the  $Oy$  direction is given by

$$\tau = \mu \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right). \quad (1.2)$$

Using (1.2), find the  $x$ -component of force exerted by the flow (1.1) on the length  $0 < x < L$  of the upper side of the boundary  $y = 0$ . Show that your result is dimensionally correct.\*

Solution

(a) At  $x = 0$ ,  $v_y = 0$ ; at  $y = 0$ ,  $v_x = 0$ .

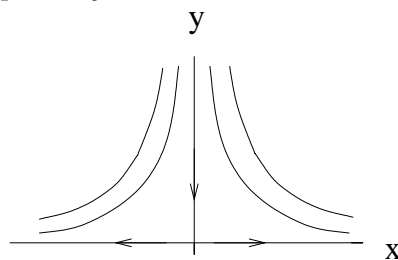
(b)

$$\frac{dx}{v_x} = \frac{dy}{v_y} \Rightarrow \frac{dx}{kyx^2} = -\frac{dy}{kxy^2}.$$

By cancelling a common factor of  $kxy$ ,

$$\frac{dx}{x} = -\frac{dy}{y} \Rightarrow ydx + xdy = 0 \Rightarrow d(xy) = 0$$

Streamlines are rectangular hyperbolae  $xy = \text{const.}$




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\* Because this is a plane flow, your answer will have dimensions of force per unit length.

(c) Method 1: Direct calculation, using definition of fluid acceleration at point  $(x, y)$  as the acceleration of the fluid particle currently at that point.

Let  $X(t); Y(t)$  be the coordinates of a fluid particle. The velocity of this particle is, by (1.1).

$$X = kY^2; \quad Y = kXY^2; \quad (1:3a; b)$$

Its acceleration has the  $x$  component

$$\ddot{X}(t) = k \frac{d}{dt} (2XY \dot{X} + X^2 \dot{Y}) = k^2 (2X^3 Y^2 + X^3 Y^2 \dot{g}) = k^2 X^3 Y^2; \quad (1:4a; b; d)$$

Eq.(1.4a) follows by using the product rule; (1.4b) follows from (1.4a) by substituting for  $X, Y$ .

The  $y$  component is

$$\ddot{Y}(t) = k \frac{d}{dt} (XY^2 + 2XY \dot{Y}) = k^2 (X^2 Y^3 + 2X^2 Y^3) = k^2 X^2 Y^3; \quad (1:5a; b; d)$$

The acceleration of the particle is :

$$a = k^2 X^2 Y^2 (iX + jY); \quad (1:6)$$

at point  $Xi + Yj$ . Because this holds for all  $X$  and  $Y$ , the acceleration is given in the spatial description by

$$a(x; t) = k^2 x^2 y^2 (ix + jy);$$

Method 2: Equivalent procedure, expressed in terms of the material derivative.

$$\begin{aligned} a &= \frac{dV}{dt} = \frac{\partial}{\partial t} + (V \cdot \nabla) V; \\ &= \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \\ &= 0 + kyx^2 \frac{\partial}{\partial x} + kxy^2 \frac{\partial}{\partial y} \\ &= kyx^2 (2kxyi + ky^2j) + kxy^2 (kx^2i + 2kxyj) \\ &= k^2 (x^3 y^2 i + x^2 y^3 j); \end{aligned}$$

as by method 1.

Sketch: Must show that  $a \cdot k \cdot r$ .

(d) For the flow (1.1)

$$\frac{\partial Y}{\partial y} = kx^2; \quad \frac{\partial Y}{\partial x} = ky^2$$

On  $y = 0$

$$= kx^2;$$

Resultant force in  $x$  direction on strip  $0 < x < L$  :

$$F_x = \int_0^L dx = \frac{1}{3} kL^3$$

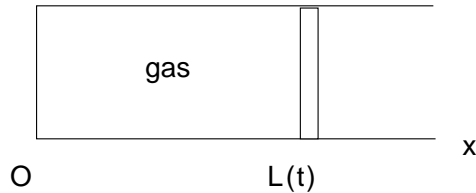
Dimensions:  $[k] = L^{-2} T^{-1}$  (from expression for  $V$ ),  $[ ] = ML^{-1} T^{-1}$  )  $[kL^3] = MT^{-2} = [F=L]$ , dimensionally consistent.

2. (100) A fixed mass of gas is compressed by pushing the piston inwards: the length  $L(t)$  of the gas column decreases with time. The flow is one-dimensional with acceleration  $a = a_x i$ ;  $a_x = \ddot{L}$  at  $x=L$ : where  $\ddot{L} = d^2L/dt^2$ . Because  $a_x \ll 0$ , the Euler equation  $a_x = -\frac{\partial p}{\partial x}$  (negligible gravity) requires there to be a pressure gradient. As a result, the density varies in  $x$  from its value  $\rho_0(t)$  at  $x=0$  to  $\rho_L(t)$  at  $x=L$ .

(a) If  $\rho_L \approx \rho_0$ , density can, to a first approximation, be taken as uniform in  $x$ . Assuming this to be so, find  $p(x;t)$  as a function of  $p_0(t) = p(0;t)$ ,  $\rho_0(t)$ ,  $\ddot{L}$ ,  $L$  and  $x$ . Show that your answer is dimensionally correct.

(b) Using the result of part (a), and Taylor's theorem, find  $\rho_L - \rho_0$  as a function of  $\rho_0$ ,  $L$ ,  $\ddot{L}$  and the (isentropic) bulk modulus.  $K_S = \rho \frac{\partial p}{\partial \rho}$ :

(c) Estimate  $(\rho_L - \rho_0)/\rho_0$  for a car engine idling with angular velocity  $\omega = 10^2$  rad/s,  $\rho_0 = 1$  kg/m<sup>3</sup>,  $K_S = 10^5$  Pa when  $L = 0.1$  m: assume that  $\ddot{L} = \omega^2 L$ .



Solution

(a) Euler equation:  $\frac{\partial p}{\partial x} = -\rho_0 \ddot{L}$

$$p(x;t) = p(0;t) - \rho_0 \ddot{L} \frac{x^2}{2L} \quad (2:1)$$

Dimensions

L.H.S. :  $[p] = FL^{-2} = ML^{-1}T^{-2}$ .

R.H.S. :  $[\rho_0 \ddot{L} x^2/L] = ML^{-3}L^3T^{-2}L^{-1} = ML^{-1}T^{-2}$ . Consistent.

(b) Taylor's theorem:

$$\rho_L(t) = \rho_0(t) + \frac{1}{2} \rho_0(t) \frac{\partial^2 \rho}{\partial x^2} L^2 + \dots \quad (2:2)$$

$$\frac{\rho_L - \rho_0}{\rho_0} = \frac{\rho_L}{\rho_0} - 1 = \frac{\rho_L}{\rho_0} - \frac{p_L}{p_0} \quad (2:3)$$

By substituting into (2.3) the result of evaluating (2.1) at  $x = L$ ,

$$\frac{\rho_L - \rho_0}{\rho_0} = -\frac{\rho_0 \ddot{L} L^2}{2(K_S)_0} \quad (2:4)$$

(c) For the numbers given,

$$\frac{\rho_L - \rho_0}{\rho_0} = -\rho_0 \frac{\omega^2 L^2}{2(K_S)_0} = -5 \times 10^{-4}$$