OVERALL
Average = 130
SD = 40

UNIVERSITY OF CALIFORNIA, BERKELEY
MECHANICAL ENGINEERING
ME106 Fluid Mechanics NAME **KEY**
1st Test, S19 *Prof S. Morris*

1. (100) In a certain plane flow, the fluid velocity $\mathbf{V} = v_x\mathbf{i} + v_y\mathbf{j}$ is given by

$$v_x = kyx^2, \qquad v_y = -kxy^2, \tag{1.1}$$

where $k > 0$ is constant.

10 (a) Show that (1.1) satisfies the no–slip condition on $x = 0$, and also on $y = 0$.

30 (b) Find, and sketch, the streamlines.

30 (c) Calculate the components a_x and a_y of the fluid acceleration. On your sketch in part (b), show the position vector $\mathbf{r}$ and the fluid acceleration $\mathbf{a}$.

30 (d) In an arbitrary flow of a Newtonian fluid, the shear stress τ exerted by the fluid in the x–direction on a surface whose normal is in the Oy direction is given by

$$\tau = \mu\Big(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\Big). \tag{1.2}$$

Using (1.2), find the x–component of force exerted by the flow (1.1) on the length $0 < x < L$ of the upper side of the boundary $y = 0$. Show that your result is dimensionally correct.*

Solution

Average = 75
SD = 20

(a) At $x = 0$, $v_y = 0$; at $y = 0$, $v_x = 0$.

(b)

+5 general eqn, +5 plug in V

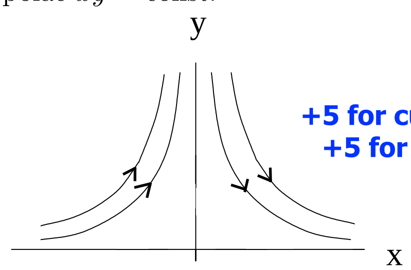
$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} \quad\Rightarrow\quad \frac{\mathrm{d}x}{kyx^2} = -\frac{\mathrm{d}y}{kxy^2}.$$

By cancelling a common factor of kxy,

+10 simplify and solve

$$\frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}y}{y} \quad\Rightarrow\quad y\mathrm{d}x + x\mathrm{d}y = 0 \quad\Rightarrow\quad \mathrm{d}(xy) = 0$$

Streamlines are rectangular hyperpolae $xy = \text{const}$.



+5 for curve shape
+5 for direction

* Because this is a plane flow, your answer will have dimensions of force per unit length.

(c) Method 1. Direct calculation, using definition of fluid acceleration at point $\{x, y\}$ as the acceleration of the fluid particle currently at that point. **(See point allocation for Method 2)**

Let $X(t), Y(t)$ be the coordinates of a fluid particle. The velocity of this particle is, by (1.1).

$$\dot{X} = kYX^2, \;\; \dot{Y} = -kXY^2. \tag{1.3a, b}$$

Its acceleration has the x–component

$$\ddot{X}(t) = k\{2XY\dot{X} + X^2\dot{Y}\}, = k^2\{2X^3Y^2 - X^3Y^2\}, = k^2X^3Y^2. \tag{1.4a, b, c}$$

Eq.(1.4a) follows by using the product rule; (1.4b) follows from (1.4a) by substituting for $\dot{X}$, $\dot{Y}$.

The y–component is

$$\ddot{Y}(t) = -k\{\dot{X}Y^2 + 2XY\dot{Y}\}, = -k^2\{X^2Y^3 - 2X^2Y^3, = k^2X^2Y^3. \tag{1.5a, b, c}$$

The acceleration of the particle is :

$$\mathbf{a} = k^2X^2Y^2\{\mathbf{i}X + \mathbf{j}Y\} : \tag{1.6}$$

at point $X\mathbf{i} + Y\mathbf{j}$. Because this holds for all X and Y, the acceleration is given in the spatial description by

$$\mathbf{a}(x, t) = k^2x^2y^2\{\mathbf{i}x + \mathbf{j}y\}.$$

Method 2. Equivalent procedure, expressed in terms of the material derivative.

$$\begin{aligned} \mathbf{a} = \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} =& \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}, \qquad \textbf{+5 general equation} \\ =& \frac{\partial \mathbf{V}}{\partial t} + v_x \frac{\partial \mathbf{V}}{\partial x} + v_y \frac{\partial \mathbf{V}}{\partial y} + v_z \frac{\partial \mathbf{V}}{\partial z} \quad \textbf{+10 expanded convective term} \\ =& 0 + kyx^2 \frac{\partial \mathbf{V}}{\partial x} - kxy^2 \frac{\partial \mathbf{V}}{\partial y} \\ =& kyx^2\{2kxy\mathbf{i} - ky^2\mathbf{j}\} - kxy^2\{kx^2\mathbf{i} - 2kxy\mathbf{j}\} \\ =& k^2\{x^3y^2\mathbf{i} + x^2y^3\mathbf{j}\}, \qquad \textbf{+5 correct answer} \end{aligned}$$

as by method 1.

Sketch. Must show that $\mathbf{a} \parallel \mathbf{r}$. **+10**

(d) For the flow (1.1)

$$\frac{\partial v_x}{\partial y} = kx^2, \;\; \frac{\partial v_y}{\partial x} = -ky^2$$

On $y = 0$

$$\tau = \mu k x^2. \qquad \textbf{+10 shear (+5/10 is not evaluated at y=0)}$$

Resultant force in x–direction on strip $0 < x < L$:

$$F_x = \int_0^L \tau \,\mathrm{d}x = \frac{1}{3}\mu k L^3 \qquad \textbf{+10 force}$$

Dimensions: $[k] = L^{-2}T^{-1}$ (from expression for $\mathbf{V}$), $[\mu] = ML^{-1}T^{-1} \;\; \Rightarrow \;\; [\mu k L^3] = MT^{-2} = [F/L]$. dimensionally consistent.

+10 dimensions

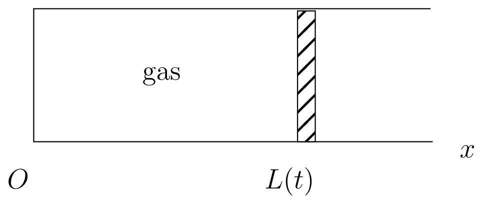
Average = 60
SD = 30

2. (100) A fixed mass of gas is compressed by pushing the piston inwards: the length $L(t)$ of the gas column decreases with time. The flow is one–dimensional with acceleration $\mathbf{a} = a_x\mathbf{i}$; $a_x = \ddot{L}\,x/L$. where $\ddot{L} = \mathrm{d}^2L/\mathrm{d}t^2$. Because $a_x \neq 0$, the Euler equation $\rho a_x = -\frac{\partial p}{\partial x}$ (negligible gravity) requires there to be a pressure gradient. As a result, the density ρ varies in x from its value $\rho_0(t)$ at $x = 0$ to $\rho_L(t)$ at $x = L$.

40 (a) If $|\rho_L - \rho_0| \ll \rho_0$, density can, to a first approximation, be taken as uniform in x. Assuming this to be so, find $p(x,t)$ as a function of $p_0(t) = p(0,t)$, $\rho_0(t)$, $\ddot{L}$, L and x. Show that your answer is dimensionally correct.

50 (b) Using the result of part (a), and Taylor's theorem, find $\rho_L - \rho_0$ as a function of ρ_0, L, $\ddot{L}$ and the (isentropic) bulk modulus. $K_S = \rho\left(\frac{\partial p}{\partial \rho}\right)_S$.

10 (c) Estimate $(\rho_L - \rho_0)/\rho_0$ for a car engine idling with angular velocity $\omega = 10^2$ rad/s, $\rho_0 = 1$ kg/m^3, $K_S = 10^5$ Pa when $L = 0.1$ m: assume that $\ddot{L} = \omega^2 L$.



Solution

(a) Euler equation: $\frac{\partial p}{\partial x} = -\rho(0,t)\frac{\ddot{L}}{L}x$,

+30 plug in for a_x
integrate
evaluate at bounds
constant of p(0,t)

$$\Rightarrow \quad p(x,t) = p(0,t) - \rho(0,t)\frac{\ddot{L}}{2L}x^2 \qquad (2.1)$$

Dimensions

L.H.S. : $[p] = FL^{-2} = ML^{-1}T^{-2}$.

R.H.S. : $[\rho\ddot{L}x^2/L] = ML^{-3}L^3T^{-2}L^{-1}, = ML^{-1}T^{-2}$. Consistent. **+10**

(b) Taylor's theorem:

\rho and p flipped

+10 Taylor's or some other method to get this eqn

$$\rho(L,t) = \rho(0,t) + \{p_L(t) - p_0(t)\}\left(\frac{\partial p}{\partial \rho}\right)_S\bigg|_0 + h.o.t. \qquad (2.2)$$

$$\Rightarrow \quad \frac{\rho_L - \rho_0}{\rho_0} = \frac{p_L - p_0}{(K_S)_0}. \qquad (2.3)$$

+20

By substituting into (2.3) the result of evaluating (2.1) at $x = L$,

+20 plug in and evaluate

$$\frac{\rho_L - \rho_0}{\rho_0} = -\rho_0\frac{L\ddot{L}}{2(K_S)_0}. \qquad (2.4)$$

(c) For the numbers given,

$$\left|\frac{\rho_L - \rho_0}{\rho_0}\right| = \rho_0\omega^2\frac{L^2}{2(K_S)_0}, = 5 \times 10^{-4}.$$

+10