Final Examination
Thursday, August 9, 2018
9:30 am to 12:00 pm
3109 Etcheverry Hall

Closed Books and Closed Notes
For Full Credit Answer All Five Questions
Question 1 is worth 10 Points
Questions 2 and 4 are worth 20 Points Each
Questions 3 and 5 are worth 30 Points Each

Useful Formulae

For the corotational basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z = \mathbf{E}_z\}$ shown in the figures

$$\begin{aligned}\mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x.\end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

The expression for the inverse of the following matrix will be useful in answering one of the questions:

$$\begin{bmatrix} m_1 & m_1 \\ m_1 & m_1 + m_2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{m_1+m_2}{m_1 m_2} & -\frac{1}{m_2} \\ -\frac{1}{m_2} & \frac{1}{m_2} \end{bmatrix}. \tag{3}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{4}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{5}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

A Rigid Body Tumbling through the Air (10 Points)

As shown in Figure 1, a slender rigid rod of length ℓ, mass m, and moment of inertia $I_{zz} = \frac{m\ell^2}{12}$ is thrown into the air. A vertical gravitational force $-mg\mathbf{E}_y$ acts on the rod and drag is ignored. The position vector of the center of mass C of the rod is described using a set of Cartesian coordinates,

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z, \tag{6}$$

and the angular velocity vector of the rod has the representation:

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z. \tag{7}$$

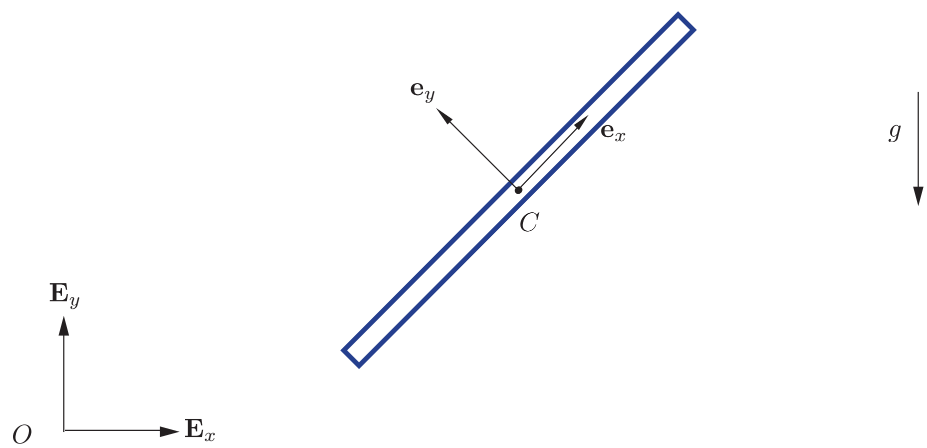

Figure 1: *A slender rod moving through the air.*

Suppose the rod is thrown into the air with the following initial conditions:

$$\bar{\mathbf{x}}\,(t=0) = x_0\mathbf{E}_x + y_0\mathbf{E}_y, \qquad \bar{\mathbf{v}}\,(t=0) = v_{0_x}\mathbf{E}_x + v_{0_y}\mathbf{E}_y, \qquad \theta\,(t=0) = \theta_0, \qquad \dot{\theta}\,(t=0) = \omega_0. \tag{8}$$

Determine the resulting motion $\bar{\mathbf{x}}(t)$ and $\theta(t)$ of the rod.

Question 2

Tipping Points *(20 Points)*

As shown in Figure 2, a crate is being moved up and down an incline with the help of a force $P_0\mathbf{E}_x$. The crate is modeled as a rigid body of mass m and moment of inertia I_{zz} relative to its center of mass C which has two contact points A and B with the incline. During its motion, a vertical gravitational force acts on the crate and the center of mass C of the crate has a position vector

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y_0\mathbf{E}_y, \tag{9}$$

where y_0 is a constant.

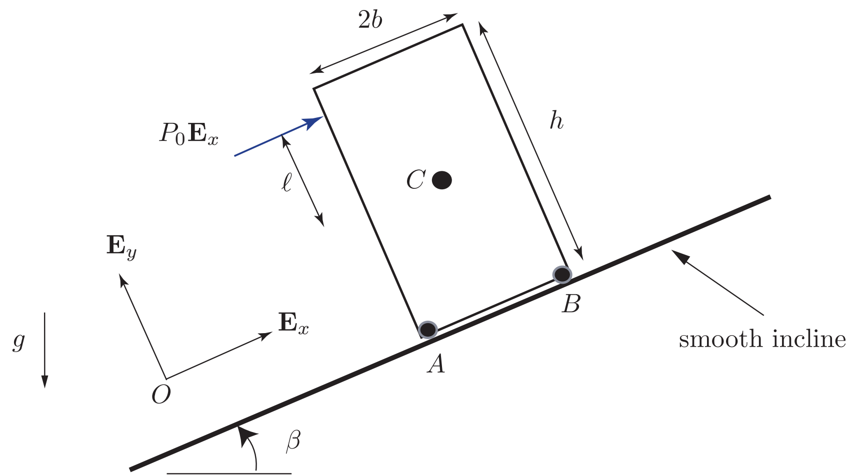

Figure 2: *A crate being pushed up an incline by a constant force $P_0\mathbf{E}_x$.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T for the rigid body assuming that it is not rotating.

(b) *(5 Points)* Assuming that both A and B are in contact with the incline, draw a freebody diagram of the rigid body.

(c) *(5 Points)* Using balances of linear and angular momentum, assuming that both A and B are in contact with the incline show that the normal forces acting at A and B are, respectively,

$$\mathbf{N}_A = \left(\frac{mg}{2}\cos(\beta) - ?\right)\mathbf{E}_y, \qquad \mathbf{N}_B = \left(\frac{mg}{2}\cos(\beta) + ?\right)\mathbf{E}_y, \tag{10}$$

and

$$m\ddot{x} = P_0 - mg\sin(\beta). \tag{11}$$

For full credit supply the missing terms.

(d) *(5 Points)* To prevent the crate from tipping over, show that P_0 must be limited as follows:

$$-\frac{b}{\ell}\cos(\beta) \leq \frac{P_0}{mg} \leq \frac{b}{\ell}\cos(\beta). \tag{12}$$

Question 3

A Suspended Rigid Body

(30 Points)

As shown in Figure 3, a thin uniform rigid plate of mass m_1, width $2w$ and breadth $2b$ is pinjointed at O. A particle of mass m_2 is rigidly attached to the plate at the location shown in the figure. During the motion of the composite rigid body of mass $m_1 + m_2$, a vertical gravitational force also acts on the rigid body along with a force $\mathbf{P}$ which has a constant magnitude P_0. The direction of the force $\mathbf{P}$ is determined by the cable that is attached to a point S on the plate. The position vectors of the center of mass C_1 of the plate and the points S and A relative to the point O, and the angular momentum of the plate relative to C_1 have the representations

$$\bar{\mathbf{x}}_1 = w\mathbf{e}_x - b\mathbf{e}_y, \qquad \mathbf{x}_S = 2w\mathbf{e}_x, \qquad \mathbf{x}_A = 2w\mathbf{E}_y, \qquad \mathbf{H}_1 = \left(I_{zz} = \frac{m_1}{3}\left(w^2 + b^2\right)\right)\dot{\theta}\mathbf{E}_z. \quad (13)$$

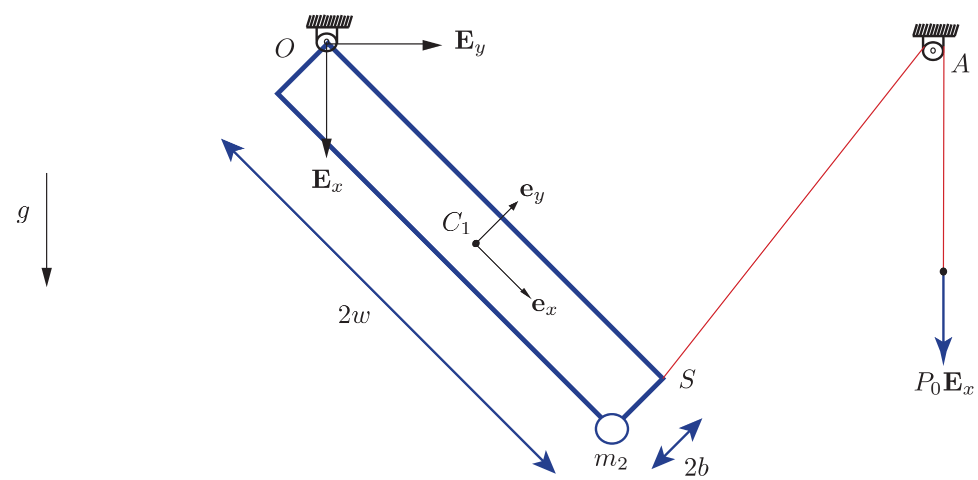

Figure 3: *A rigid body of mass $m_1 + m_2$ is free to rotate about O and is subject to a gravitational force.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T of the rigid body of mass $m_1 + m_2$ when it is rotating about O.

(b) *(5 Points)* Draw a free–body diagram of the rigid body when it is rotating about O.

(c) *(5 Points)* Show that the following differential equation governs θ when the body is rotating about O:

$$\left(\frac{4m_1}{3}\left(w^2 + b^2\right) + 4m_2\left(w^2 + b^2\right)\right)\ddot{\theta} = ????g + ?????P_0. \quad (14)$$

For full credit, supply the missing terms (some of which may be negative).

(d) *(5 Points)* Prove that the force applied by the cable to the rigid body is conservative with a potential energy

$$U_{\mathbf{P}} = P_0 \left\| \mathbf{x}_A - \mathbf{x}_S \right\|. \quad (15)$$

(e) *(5 Points)* Starting from the work-energy theorem (5), prove that the total energy E of the rigid body is conserved. For full credit, supply an expression for the total energy E.

(f) *(5 Points)* Suppose that the rigid body is released from rest with $\theta = 0$. Determine the minimum value of the force P_0 needed to raise the body to $\theta = \frac{\pi}{2}$.

Question 4

A Particle in a Slotted Disk (20 Points)

As shown in Figure 4, a particle of mass m_1 is attached by a spring to a circular disk of mass m_2. The particle is free to move in a smooth grove milled on the surface of the circular disk. The disk is fixed at its center point O and is free to rotate about a vertical $\mathbf{E}_z$ axis.

The position vector of the particle relative to the center O of the disk is

$$\mathbf{x}_1 = x\mathbf{e}_x. \tag{16}$$

The moment of inertia of the disk is $I_{O_{zz}}$ and the stiffness and unstretched length of the spring are K and ℓ_0, respectively.

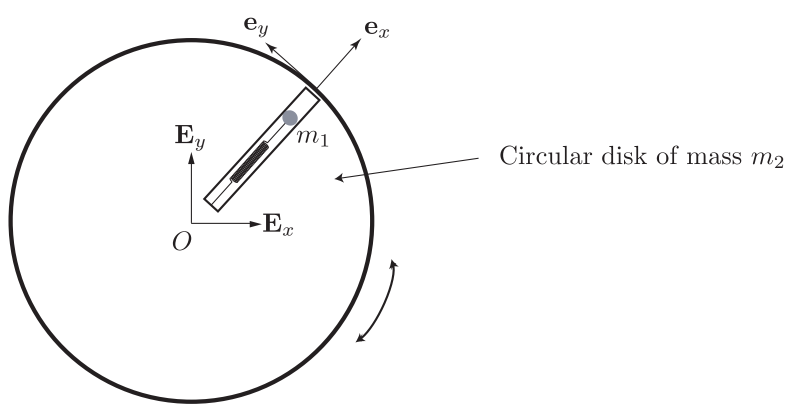

Figure 4: *A particle of mass m_1 is free to move in a smooth slot on a circular disk of mass m_2.*

(a) *(5 Points)* Show that the kinetic energy and angular momentum relative to O of the particle-disk system is

$$\begin{aligned} \mathbf{H}_O &= \left(I_{O_{zz}} + m_1 x^2\right)\dot{\theta}\mathbf{E}_z, \\ T &= \frac{m_1}{2}\dot{x}^2 + \frac{1}{2}\left(I_{O_{zz}} + m_1 x^2\right)\dot{\theta}^2. \end{aligned} \tag{17}$$

(b) *(5 Points)* Draw a free-body diagram of the particle and a free-body diagram of the circular disk. For full credit, provide a clear expression for the spring force.

(c) *(5 Points)* Show that the equations governing the motion of the disk-particle system are

$$I_{O_{zz}}\ddot{\theta} + ?? + ??? = 0, \qquad m_1\left(\ddot{x} - x\dot{\theta}^2\right) = ????. \tag{18}$$

For full credit supply the missing terms in these equations.

(d) *(5 Points)* Give an expression for the total energy E of the system and verify either with the help of (18) or using a work-energy theorem, that $\dot{E} = 0$.

Question 5

Sliding of a Rigid Body (30 Points)

As shown in Figure 5, a circular cylinder of radius R, mass m_1, and moment of inertia (relative to its center of mass C_1) I_{zz} is free to move atop a cart of mass m_2. A constant force $F_0\mathbf{E}_x$ acts on the cart. The position vectors of the centers of mass C_1 and C_2 have the representations

$$\bar{\mathbf{x}}_1 = (x_1 + x_2)\,\mathbf{E}_x + y_0\mathbf{E}_y, \qquad \bar{\mathbf{x}}_2 = x_2\mathbf{E}_x, \tag{19}$$

where y_0 is a constant. The point P is the instantaneous point of contact of the cylinder and the cart.

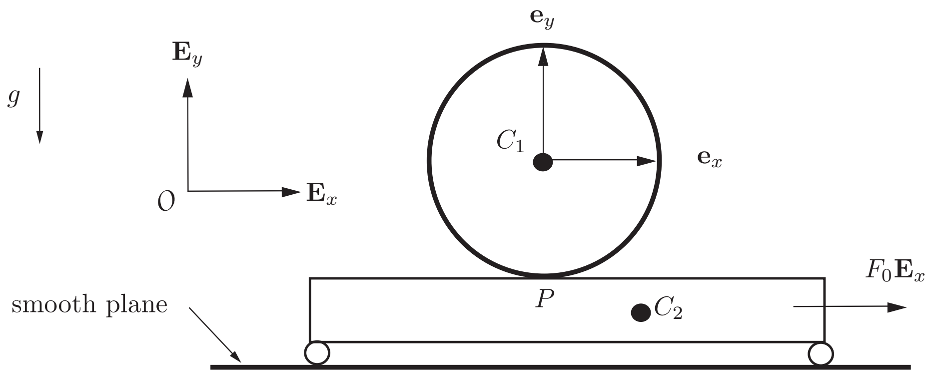

Figure 5: *A rigid cylinder of mass m_1 and radius R moving on a cart. A constant force $F_0\mathbf{E}_x$ acts on the cart.*

(a) *(5 Points)* With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the cylinder, show that slip speed v_{rel} of the cylinder relative to the cart is

$$v_{\text{rel}} = \dot{x}_1 + R\dot{\theta}, \tag{20}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the cylinder.

(b) *(5 Points)* Draw free-body diagrams of the cylinder and the cart. Assume that the cylinder is sliding on the cart. Provide a clear expression for the friction force $\mathbf{F}_f$ acting on the cylinder.

(c) *(10 Points)* Assume that the cylinder is sliding on the cart. Show that the equations of motion of the system can be expressed in the following fashion:

$$\begin{aligned}\begin{bmatrix} m_1 & m_1 \\ m_1 & m_1 + m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\mu_k m_1 g?? \\ ??? \end{bmatrix}, \\ I_{zz}\ddot{\theta} &= -\mu_k m_1 g R??. \end{aligned} \tag{21}$$

For full credit supply the missing terms.

(d) *(5 Points)* Show that the slip speed of the cylinder relative to the cart is governed by the equation

$$\dot{v}_{\text{rel}} = -\frac{F_0}{m_2} - \mu_k m_1 g \left(\frac{R^2}{I_{zz}} + \frac{m_1 + m_2}{m_1 m_2} \right) \frac{v_{\text{rel}}}{\left| v_{\text{rel}} \right|} \tag{22}$$

(e) *(5 Points)* With the help of a work-energy theorem, show that the total energy E of the system decreases with time. In your solution, provide a clear expression for E