Final Examination
Tuesday, May 8, 2018
8:00 am to 11:00 am
141 McCone Hall and 105 Northgate

Closed Books and Closed Notes
For Full Credit Answer All Four Questions

Useful Formulae

For the corotational basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z = \mathbf{E}_z\}$ shown in the figures

$$\begin{aligned} \mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x. \end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{3}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{4}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

A Suspended Plate

(25 Points)

As shown in Figure 1, a thin uniform rigid plate of mass m, width $2w$ and breadth $2b$ is pinjointed at O. During the motion of the plate, a vertical gravitational force also acts on the rigid body along with a force supplied by a spring attached to a point S on the plate. The unstretched length ℓ_0 of the spring is identical to the distance $\|\mathbf{x}_B - \mathbf{x}_A\|$. The position vectors of the center of mass $\bar{X}$ of the rigid body and the points S and A relative to the point O, and the angular momentum of the rigid body relative to $\bar{X}$ have the representations

$$\bar{\mathbf{x}} = w\mathbf{e}_x - b\mathbf{e}_y, \qquad \mathbf{x}_S = 2w\mathbf{e}_x, \qquad \mathbf{x}_A = 2w\mathbf{E}_y, \qquad \mathbf{H} = \left(I_{zz} = \frac{m}{3}\left(w^2 + b^2\right)\right)\dot{\theta}\mathbf{E}_z. \tag{5}$$

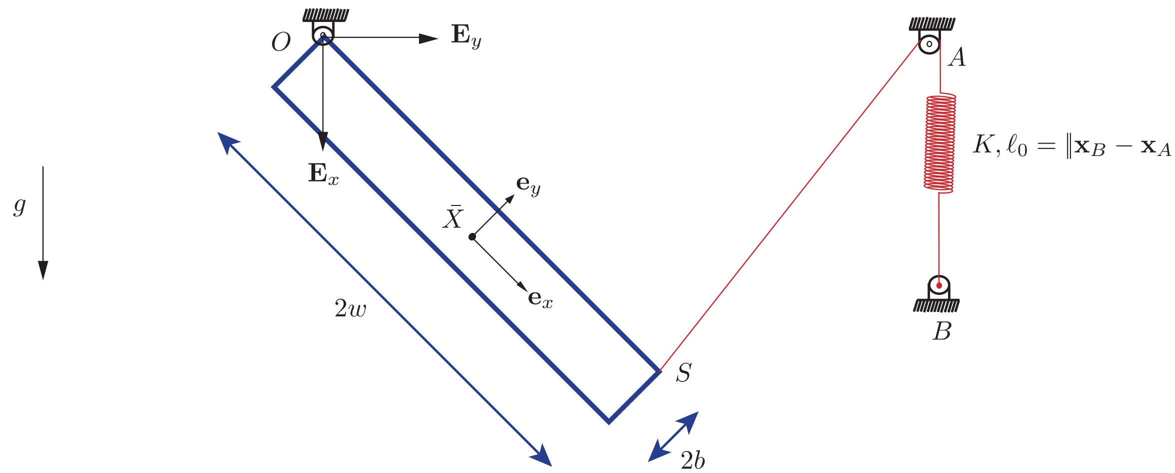

Figure 1: *A rigid body of mass m is free to rotate about O and is subject to a gravitational and spring forces.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T of the rigid body when it is rotating about O.

(b) *(5 Points)* Draw a free–body diagram of the rigid body when it is rotating about O. Verify that the extension ϵ of the spring is $\epsilon = 2w\sqrt{2\left(1 - \sin(\theta)\right)}$.

(c) *(5 Points)* Show that the following differential equation governs θ when the body is rotating about O:

$$\frac{4m}{3}\left(w^2 + b^2\right)\ddot{\theta} = mg\left(b? + w??\right) + K??? \tag{6}$$

For full credit, supply the missing terms (some of which may be negative).

(d) *(5 Points)* Starting from the work-energy theorem (4), prove that the total energy E of the rigid body is conserved. For full credit, supply an expression for the total energy E.

(e) *(5 Points)* Suppose that the rigid body is released from rest with $\theta = \frac{\pi}{2}$. Show that provided the spring is sufficiently weak, i.e.,

$$\frac{K}{mg} < \frac{w - b}{4w^2}, \tag{7}$$

then the rigid body will pass through $\theta = 0$. Determine the speed $\dot{\theta}$ that it would have at this instant.

Question 2

Loading a Wheelbarrow (20 Points)

As shown in Figure 2, a wheelbarrow and its load are being moved on level ground by an applied force $\mathbf{F}_0 = F_{0_x}\mathbf{E}_x + F_{0_y}\mathbf{E}_y$ acting at the handle H. The wheelbarrow and its load has a combined mass m and a moment of inertia I_{zz} relative to the center of mass $\bar{X}$. The position vectors of $\bar{X}$, H, and the point of contact P of the wheel with the ground are

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y\mathbf{E}_y, \qquad \mathbf{x}_H = \bar{\mathbf{x}} - \ell_1\mathbf{e}_x + \ell_2\mathbf{e}_y, \qquad \mathbf{x}_P = \bar{\mathbf{x}} - h_1\mathbf{e}_x - h_2\mathbf{e}_y. \tag{8}$$

The mass and inertia of the wheel can be ignored and the wheelbarrow and its load are modeled as a single rigid body.

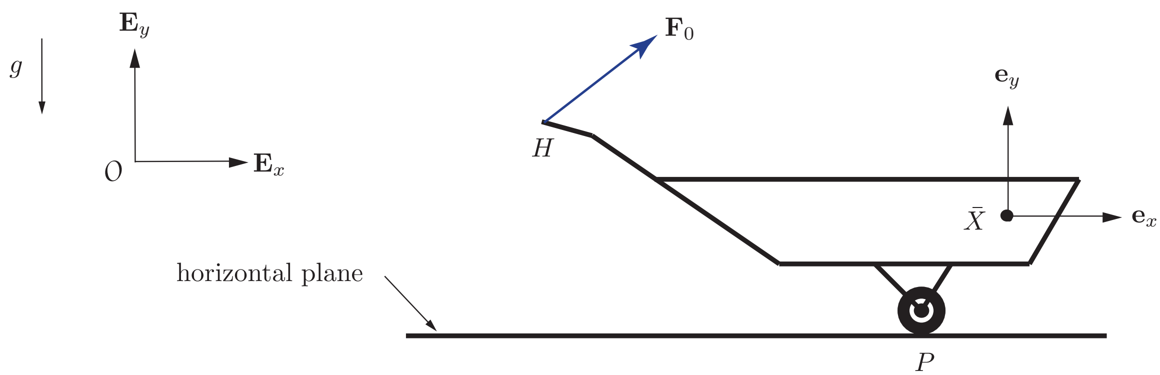

Figure 2: *A wheelbarrow and its load moving on horizontal surface. A force $\mathbf{F}_0$ is applied at the handle H of the wheelbarrow.*

(a) *(2 Points)* Assuming that y is constant, the wheelbarrow is not rotating, and $\theta = 0$, establish expressions for the acceleration $\bar{\mathbf{a}}$ of the center of mass and the angular momentum of the rigid body relative to $\bar{X}$.

(b) *(4 Points)* Draw a free-body diagram of the rigid body assuming that a normal force and a traction force act at the point P.

(c) *(10 Points)* Using balances of linear and angular momentum, show that

$$\frac{\ddot{x}}{g} = \left(1 + \frac{\ell_2}{h_2}\right)\frac{F_{0_x}}{mg} + \left(\frac{\ell_1}{h_2}\right)\frac{F_{0_y}}{mg} + \frac{h_1}{h_2}\left(1 - \frac{F_{0_y}}{mg}\right). \tag{9}$$

(d) *(4 Points)* Assuming that the wheel remains in contact with the ground, show that by arranging the load in the wheelbarrow so that the center of mass of the rigid body is ahead of P (i.e., increasing h_1 and ℓ_1 while keeping ℓ_2 and h_2 constant), then a greater acceleration of the rigid body can be achieved for a given $\mathbf{F}_0$ with $F_{0_x} > 0$ and $F_{0_y} > 0$.

Question 3

Motion of a Suspended Pendulum *(35 Points)*

As shown in Figure 3, a rigid rod of length 2ℓ, moment of inertia relative to its center of mass of I_{zz}, and mass m_1 is attached to a collar of mass m_2 at a point A by a pin joint. The rod is free to rotate about A and is restrained by a torsional spring of stiffness K_T and moment $-K_T\theta\mathbf{E}_z$. A vertical gravitational force $m_1 g\mathbf{E}_x$ also acts on the rod. A constant force $P\mathbf{E}_y$ acts on the collar of mass m_2 as it slides back and forth on a smooth guide rail. You can assume that the center of mass of the collar and A are coincident: $\bar{\mathbf{x}}_2 = \mathbf{x}_A$.

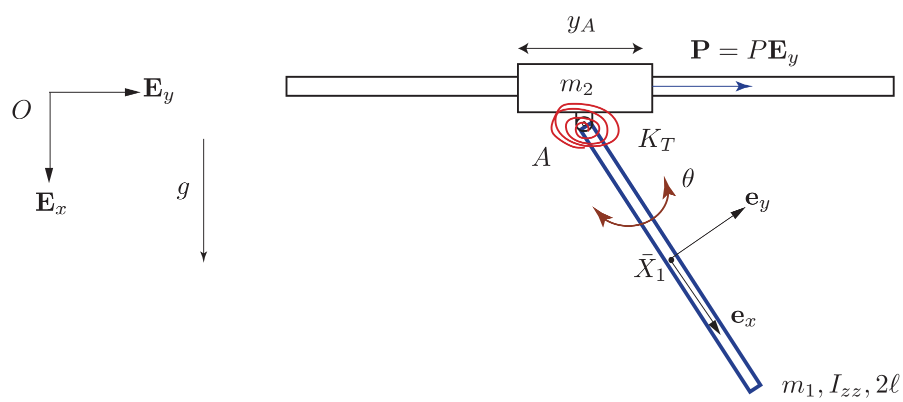

Figure 3: *A rigid body of mass m_1 that is pin-jointed to a collar of mass m_2 that is free to move on a smooth track. Note that the gravitational force in this system is in the $\mathbf{E}_x$ direction.*

(a) *(8 Points)* Starting from the following representation for the position vector of the center of mass $\bar{X}_1$ of the rod of mass m_1 relative to A,

$$\bar{\mathbf{x}}_1 - \mathbf{x}_A = \ell\mathbf{e}_x, \qquad \text{where} \qquad \bar{\mathbf{x}}_2 = \mathbf{x}_A = y_A\mathbf{E}_y, \tag{10}$$

establish expressions for the linear momentum $\mathbf{G}_1$, angular momentum $\mathbf{H}_{1_A}$ and kinetic energy T_1 of the rigid body of mass m_1.

(b) *(7 Points)* Draw freebody diagrams of the rigid body of mass m_1 and the rigid body of mass m_2.

(c) *(3 Points)* Show that the reaction force at A that act on the rigid body of mass m_1 is

$$\mathbf{R}_A = -m_1 g\mathbf{E}_x + m_1\ddot{y}_A\mathbf{E}_y + m_1\ell\left(\ddot{\theta}\mathbf{e}_y - \dot{\theta}^2\mathbf{e}_x\right). \tag{11}$$

(d) *(7 Points)* Show that the equation of motion for the rigid body of mass m_1 is

$$\left(I_{zz}+?\right)\ddot{\theta} = m_1\ell\ddot{y}_A?? - m_1\ell g\sin(\theta) - K_T\theta. \tag{12}$$

For full credit supply the missing terms (some of which may be negative).

(e) *(5 Points)* Show that

$$(m_1 + m_2)\dot{y}_A + m_1\ell\dot{\theta}\cos(\theta) \tag{13}$$

is *not* conserved during the motion of the system.

(f) *(5 Points)* Provide an expression for the total energy E of this system and, with the help of a work-energy theorem of your choice, explain why this energy is conserved.

Question 4

A Model for a Spherical Robot (30 Points)

As shown in Figure 4, a pendulum consisting of a particle of mass m_2 and a massless rigid rod of length ℓ_0 is mounted by a pin-joint to the geometric center C of a cylindrical shell of radius R, mass m_1, and moment of inertia (relative to its center of mass $\bar{X}_1$) I_{zz}. The pendulum is free to swing in the vertical plane and the cylinder moves on a rough horizontal plane. When the cylinder rolls on the plane, the motion of pendulum enables locomotion of the cylinder. This simple mechanism is a prototype drive for earlier designs of spherical robots.

In the sequel, we assume that the rigid rod of length $2R$ attaching the pin joint to the cylindrical shell are of negligible mass, so the center of mass $\bar{X}_1$ of the cylinder and mounting rod is at its geometric center C. The center of mass $\bar{X}$ of the composite rigid body of mass $m = m_1 + m_2$ has the representation

$$\bar{\mathbf{x}} = \mathbf{x}_C + h \left(\mathbf{p}_x = \cos(\phi)\mathbf{E}_x + \sin(\phi)\mathbf{E}_y \right), \tag{14}$$

where

$$\mathbf{x}_C = x\mathbf{E}_x + y_0\mathbf{E}_y, \qquad h = \frac{m_2}{m}\ell_0, \qquad m = m_1 + m_2. \tag{15}$$

The point P is the instantaneous point of contact of the cylindrical shell and the plane.

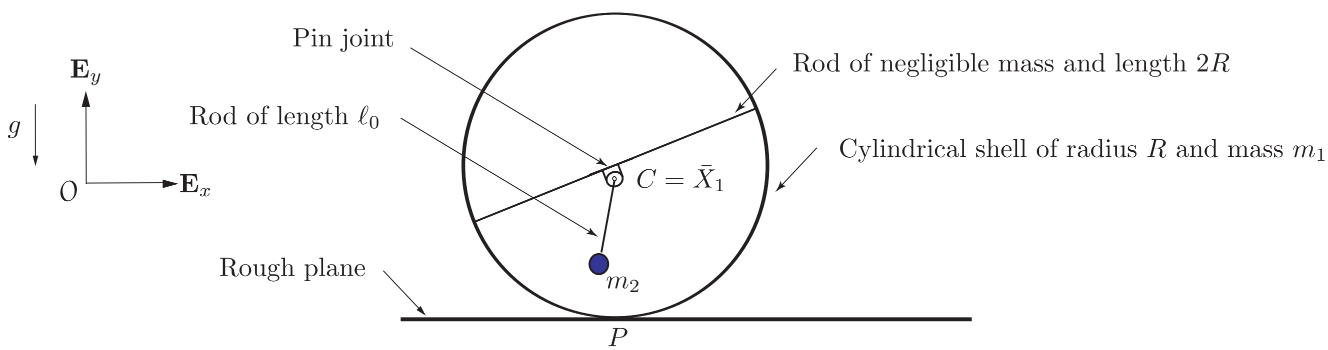

Figure 4: *A cylindrical shell of radius R rolling on a horizontal plane. The shell is driven to move by the motion of a suspended pendulum of mass m_2.*

As shown in Figure 5, a corotational basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z = \mathbf{E}_z\}$ is attached to the cylindrical shell and a corotational basis $\{\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z = \mathbf{E}_z\}$ is attached to the pendulum. We also note that $\boldsymbol{\omega}_1 = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the cylindrical shell and $\boldsymbol{\omega}_2 = \dot{\phi}\mathbf{E}_z$ is the angular velocity of the pendulum.

$$\left\{\begin{aligned} \mathbf{e}_x &= \cos(\theta)\,\mathbf{E}_x + \sin(\theta)\,\mathbf{E}_y \\ \mathbf{e}_y &= -\sin(\theta)\,\mathbf{E}_x + \cos(\theta)\,\mathbf{E}_y \end{aligned}\right\}$$

$$\left\{\begin{aligned} \mathbf{p}_x &= \cos(\phi)\,\mathbf{E}_x + \sin(\phi)\,\mathbf{E}_y \\ \mathbf{p}_y &= -\sin(\phi)\,\mathbf{E}_x + \cos(\phi)\,\mathbf{E}_y \end{aligned}\right\}$$

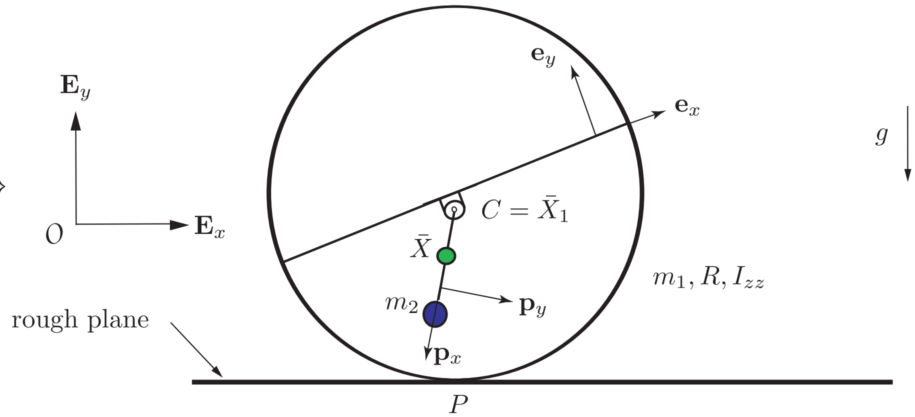

Figure 5: *A cylindrical shell of radius R rolling on a horizontal plane. The point $\bar{X}$ on the rod of length ℓ_0 is the center of mass of the system.*

(a) *(5 Points)* With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the cylindrical shell, show that slip speed v_P of the shell (and the attached massless rods) is

$$v_P = \dot{x} + R\dot{\theta}. \tag{16}$$

(b) *(10 Points)* Show that the angular momentum of the system relative to $\bar{X}$ and the kinetic energy of the system has the representation

$$\mathbf{H} = I_{zz}\dot{\theta}\mathbf{E}_z + \frac{m_1 m_2}{m_1 + m_2}\ell_0^2\dot{\phi}\mathbf{E}_z, \qquad T = \frac{1}{2}\left(I_{zz}\dot{\theta}^2 + m_1\dot{x}^2\right) + \frac{m_2}{2}\left(\dot{x}^2 + \ell_0^2\dot{\phi}^2 - 2\ell_0\dot{x}\dot{\phi}\sin(\phi)\right). \tag{17}$$

(c) *(5 Points)* Draw a free body diagram of the particle of mass m_2 and a free body diagram of the cylindrical shell and the attached massless rod. Give clear expressions for the friction force when the shell is sliding and rolling.

(d) *(5 + 5 Points)* Assume that the shell is rolling. Answer 2 of the 5 following questions (if you choose to answer more than 2 questions then the top two numerical scores will be used to compute your points total for this problem):

(i) Show that

$$\mathbf{F}_f + \mathbf{N} = (m_1 + m_2)\,\ddot{x}\mathbf{E}_x + (m_1 + m_2)\,g\mathbf{E}_y + m_2\ell_0\left(\ddot{\phi}\mathbf{p}_y - \dot{\phi}^2\mathbf{p}_x\right), \tag{18}$$

where $\mathbf{F}_f$ and $\mathbf{N}$ are the respective friction force and normal force acting at P.

(ii) Using a balance of linear momentum for the particle of mass m_2, show that

$$m_2\ell_0\ddot{\phi} + m_2R\ddot{\theta}\sin(\phi) = -m_2 g\cos(\phi). \tag{19}$$

(iii) Using a balance of angular momentum for the cylindrical shell and the attached massless rod of length $2R$, show with the help of (18) that

$$\left(I_{zz} + (m_1 + m_2)\,R^2\right)\ddot{\theta} + m_2R\ell_0\sin(\phi)\,\ddot{\phi} = -m_2\ell_0\dot{\phi}^2\cos(\phi). \tag{20}$$

(iv) If you were asked to determine the motion of the system, then which differential equations would you integrate (numerically) and what set of initial conditions would you need to perform this integration?

(v) What is the total energy E of this system? With the help of a work-energy theorem of your choice, prove that energy is conserved during a motion of the system.