Final Examination
Thursday, August 11, 2016
9:30 am to 12:00 pm
155 Donner Laboratory

Closed Books and Closed Notes
For Full Credit Answer All Four Questions

Useful Formulae

For all the corotational bases shown in the figures

$$\begin{aligned} \mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x. \end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{3}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{4}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

A Suspended Plate

(25 Points)

As shown in Figure 1, a thin uniform rigid plate of mass m, width $2w$ and breadth $2b$ is pinjointed at O. During the motion of the plate, a vertical gravitational force also acts on the rigid body along with a force supplied by a spring attached to a point S on the plate. The unstretched length ℓ_0 of the spring is identical to the distance $\|\mathbf{x}_B - \mathbf{x}_A\|$. The position vectors of the center of mass C of the rigid body and the points S and A relative to the point O, and the angular momentum of the rigid body relative to C have the representations

$$\bar{\mathbf{x}} = b\mathbf{e}_x + w\mathbf{e}_y, \qquad \mathbf{x}_S = 2w\mathbf{e}_y, \qquad \mathbf{x}_A = 2w\mathbf{E}_y, \qquad \mathbf{H} = \left(I_{zz} = \frac{m}{3}\left(w^2 + b^2\right)\right)\dot{\theta}\mathbf{E}_z. \tag{5}$$

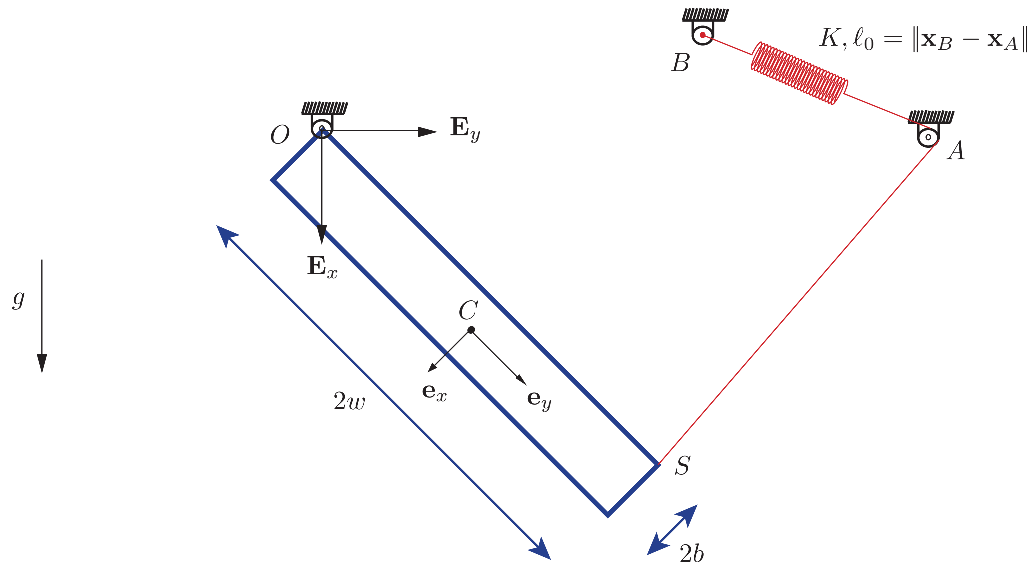

Figure 1: *A rigid body of mass m is free to rotate about O and is subject to a gravitational and spring forces.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T of the rigid body when it is rotating about O.

(b) *(5 Points)* Draw a free–body diagram of the rigid body when it is rotating about O. Verify that the extension ϵ of the spring is $\epsilon = 2w\sqrt{2\left(1 - \cos(\theta)\right)}$.

(c) *(5 Points)* Show that the following differential equation governs θ when the body is rotating about O:

$$\frac{4m}{3}\left(w^2 + b^2\right)\ddot{\theta} = -mg\left(b? + w??\right) + K??? \tag{6}$$

For full credit, supply the missing terms.

(d) *(5 Points)* Starting from the work-energy theorem (4), prove that the total energy E of the rigid body is conserved. For full credit, supply an expression for the total energy E.

(e) *(5 Points)* Outline how the reaction force $\mathbf{R}_O$ at O can be expressed as a function of energy E and the material and geometric variables, m, b, w, K, mg, and θ.

Question 2

Moving a Wheelbarrow (20 Points)

As shown in Figure 2, a wheelbarrow and its load are being moved on level ground by an applied force $\mathbf{F}_0 = F_{0_x}\mathbf{E}_x + F_{0_y}\mathbf{E}_y$ acting at the handle H. The wheelbarrow and its load has a combined mass m and a moment of inertia I_{zz} relative to the center of mass $\bar{X}$. The position vectors of $\bar{X}$, H, and the point of contact P of the wheel with the ground are

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y\mathbf{E}_y, \qquad \mathbf{x}_H = \bar{\mathbf{x}} - \ell_1\mathbf{e}_x + \ell_2\mathbf{e}_y, \qquad \mathbf{x}_P = \bar{\mathbf{x}} - h_1\mathbf{e}_x - h_2\mathbf{e}_y. \tag{7}$$

The mass and inertia of the wheel are ignored and the wheelbarrow and its load are modeled as a single rigid body.

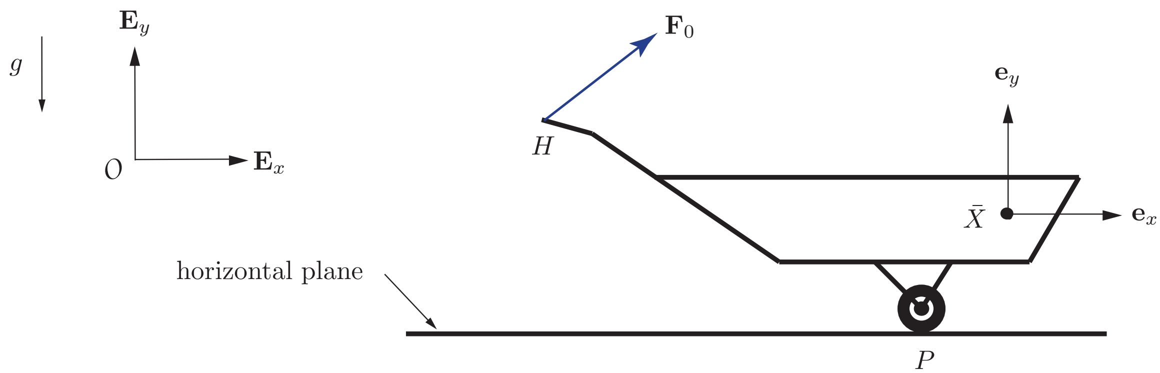

Figure 2: *A wheelbarrow and its load moving on horizontal surface. A force $\mathbf{F}_0$ is applied at the handle H of the wheelbarrow.*

(a) *(2 Points)* Assuming that y is constant, the wheelbarrow is not rotating, and $\theta = 0$, establish expressions for the acceleration $\bar{\mathbf{a}}$ of the center of mass and the angular momentum of the rigid body relative to $\bar{X}$.

(b) *(3 Points)* Draw a free-body diagram of the rigid body assuming that a normal force and a traction force act at the point P.

(c) *(10 Points)* Using balances of linear and angular momentum, show that

$$\ddot{x} = \left(1 + \frac{\ell_2}{h_2}\right)\frac{F_{0_x}}{m} + \left(\frac{\ell_1 - h_1}{h_2}\right)\frac{F_{0_y}}{m} + \frac{h_1}{h_2}g. \tag{8}$$

(d) *(5 Points)* Assuming that the wheel remains in contact with the ground, show that by arranging the load in the wheelbarrow so that the center of mass of the rigid body is ahead of P (i.e., $h_1 > 0$), then a greater acceleration of the rigid body can be achieved for a given $\mathbf{F}_0$.

Question 3

Motion of a Pendulum *(30 Points)*

As shown in Figure 3, a rigid rod of length 2ℓ, moment of inertia relative to its center of mass of I_{zz}, and mass m_1 is attached to a collar of mass m_2 at a point A by a pin joint. The rod is free to rotate about A and is restrained by a torsional spring of stiffness K_T and moment $-K_T\left(\theta + \frac{\pi}{2}\right)\mathbf{E}_z$. A vertical gravitational force $-m_1 g\mathbf{E}_y$ also acts on the rod. The collar of mass m_2 is free to slide back and forth on a smooth guide rail. You can assume that the center of mass of the collar and A are coincident: $\bar{\mathbf{x}}_2 = \mathbf{x}_A$.

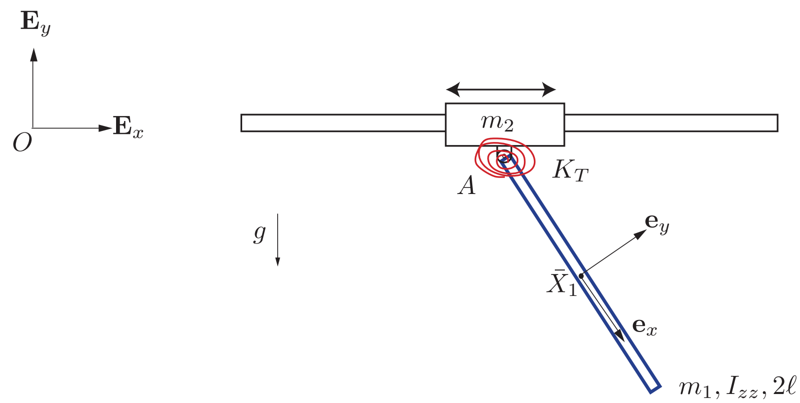

Figure 3: *A rigid body of mass m_1 that is pin-jointed to a collar of mass m_2 that is free to move on a smooth track. Note that the gravitational force in this system is in the $-\mathbf{E}_y$ direction.*

(a) *(6 Points)* Starting from the following representation for the position vector of the center of mass $\bar{X}_1$ of the rod of mass m_1 relative to A,

$$\bar{\mathbf{x}}_1 - \mathbf{x}_A = \ell\mathbf{e}_x, \qquad \text{where } \mathbf{x}_A = x_A\mathbf{E}_x = \bar{\mathbf{x}}_2, \tag{9}$$

establish expressions for the linear momentum $\mathbf{G}_1$, angular momentum $\mathbf{H}_{1_A}$ and kinetic energy T_1 of the rigid body of mass m_1.

(b) *(6 Points)* Draw freebody diagrams of the rigid body of mass m_1 and the rigid body of mass m_2.

(c) *(6 Points)* Show that the reaction force at A that act on the rigid body of mass m_1 is

$$\mathbf{R}_A = m_1 g\mathbf{E}_y + m_1\ddot{x}_A\mathbf{E}_x + m_1\ell\left(\ddot{\theta}\mathbf{e}_y - \dot{\theta}^2\mathbf{e}_x\right). \tag{10}$$

(d) *(6 Points)* Show that the equation of motion for the rigid body of mass m_1 is

$$\left(I_{zz} + ?\right)\ddot{\theta} = m_1\ell\ddot{x}_A?? - m_1\ell g\cos(\theta) - K_T\left(\theta + \frac{\pi}{2}\right). \tag{11}$$

For full credit supply the missing terms.

(e) *(6 Points)* Answer either (i) or (ii):

(i) Show that

$$(m_1 + m_2)\dot{x}_A - m_1\ell\dot{\theta}\sin(\theta) \tag{12}$$

is conserved during the motion of the system.

(ii) Provide an expression for the total energy of this system and, with the help of a work-energy theorem, explain why this energy is conserved.

Question 4

Rolling and Sliding of a Rigid Body *(25 Points)*

As shown in Figure 4, a semicircular cylinder of radius R, mass m_1, and moment of inertia (relative to its center of mass $\bar{X}_1$) I_{zz} is free to move on a rough horizontal surface. A post of negligible mass is inserted into the upper surface of the cylinder and a particle of mass m_2 is rigidly attached to the post. By varying the height h_2 of m_2 relative to C, the location of the center of mass $\bar{X}$ of the composite rigid body of mass $m = m_1 + m_2$ can be varied relative to the geometric center C of the cylinder. In the sequel, the position vectors of C and $\bar{X}$ have the following representations:

$$\mathbf{x}_C = x\mathbf{E}_x + y_0\mathbf{E}_y, \qquad \bar{\mathbf{x}} = \mathbf{x}_C + h\mathbf{e}_y, \text{ where } h = -\frac{m_1}{m}\left(\frac{4R}{3\pi}\right) + \frac{m_2}{m}h_2, \; m = m_1 + m_2, \tag{13}$$

and y_0 is a constant. The point P is the instantaneous point of contact of the cylinder and the plane, and $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity.

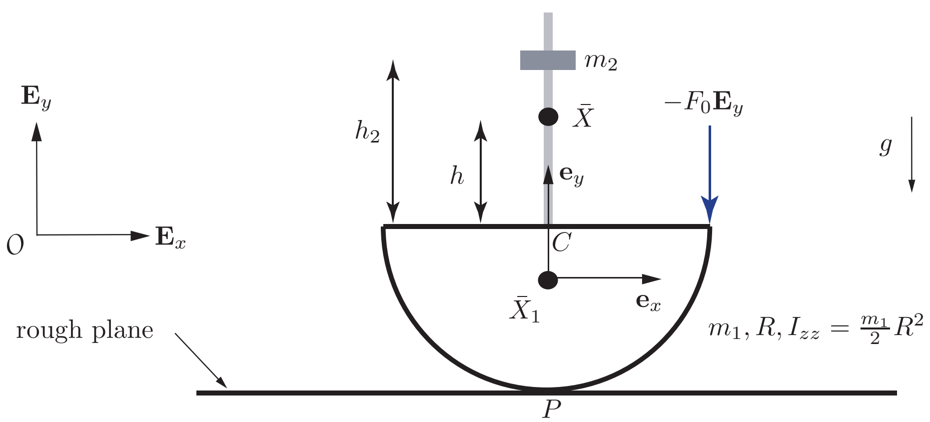

Figure 4: *A rigid semicircular cylinder of mass m_1 and radius R moving on a horizontal surface. A force $-F_0\mathbf{E}_y$ acts on the rigid body.*

(a) *(5 Points)* With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the cylinder, show that slip speed v_P of the rigid body of mass $m = m_1 + m_2$ is

$$v_P = \dot{x} + R\dot{\theta}. \tag{14}$$

(b) *(5 Points)* Draw a pair of free-body diagrams of the rigid body: one where the body is rolling and an other for when the body is slipping. For full credit, provide a clear expression for the dynamic friction force.

(c) *(5 Points)* Assume that the rigid body is rolling. Using a balance of linear momentum, show that

$$\mathbf{F}_f + \mathbf{N} = (mg + F_0)\,\mathbf{E}_y - m\ddot{\theta}\,(?\mathbf{E}_x + ??\mathbf{e}_x) - mh\dot{\theta}^2\mathbf{e}_y, \tag{15}$$

where $\mathbf{F}_f$ and $\mathbf{N}$ are the respective friction force and normal force on the rigid body. For full credit supply the missing terms in (15).

(d) *(10 Points)* With the help of (15), show that the equation governing the motion of the body can be expressed as

$$\left(\hat{I}_{zz} + mR^2 + mh^2 + 2mhR\cos(\theta)\right)\ddot{\theta} = mRh\dot{\theta}^2\sin(\theta) + mgh\sin(\theta) - F_0R\cos(\theta). \tag{16}$$

where $\hat{I}_{zz}$ is the moment of inertia of the rigid body of mass $m_1 + m_2$ about its center of mass.