Final Examination
Tuesday May 10, 2016
8:00am to 11:00 am
105 Stanley Hall

Closed Books and Closed Notes
For Full Credit Answer Five Questions of your Choice

Useful Formulae

For the corotational bases shown in the figures:

$$\begin{aligned}\mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x.\end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}, \tag{3}$$

where **a** and **b** are any pair of vectors.

You should also note that

$$|x| = +x \text{ if } x > 0 \text{ and } |x| = -x \text{ if } x < 0. \tag{4}$$

These results are useful when calculating magnitudes.

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{5}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point X_i where the force $\mathbf{F}_i$ is applied and $\mathbf{M}_p$ is a pure moment.

Question 1

Motion of a Rigid Rod

(20 Points)

As shown in Figure 1, a thin uniform rod of mass m and length 2ℓ is pin-jointed at O. One end of a spring of stiffness K and unstretched length $\ell_0 = 0$ is attached to a point B at the apex of the rod. The other end of the spring is attached to a fixed point A. During the ensuing motion, a vertical gravitational force $-mg\mathbf{E}_y$ also acts on the rigid body.

The position vectors of the center of mass C of the rigid body relative to O, the point A relative to the point O, and the point B relative to O, and the angular momentum of the rigid body relative to C have the representations

$$\bar{\mathbf{x}} = \ell \mathbf{e}_x, \qquad \mathbf{x}_B = 2\ell \mathbf{e}_x, \qquad \mathbf{x}_A = 2\ell \mathbf{E}_y, \qquad \mathbf{H} = \left(I_{zz} = \frac{m\ell^2}{3} \right) \dot{\theta} \mathbf{E}_z. \tag{6}$$

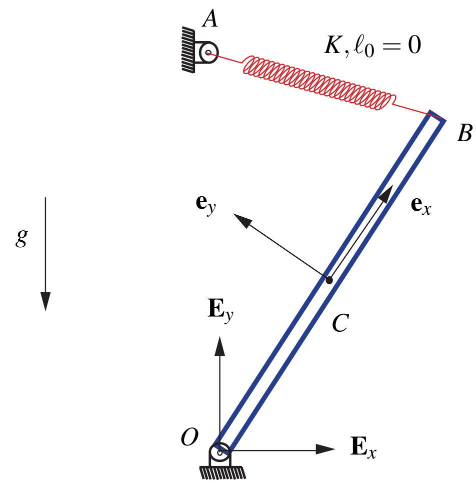

Figure 1: *A rigid body of mass m is free to rotate about O. The rigid body is subject to a gravitational force and a spring force induced by a spring of stiffness K and unstretched length $\ell_0 = 0$.*

(a) *(6 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T of the rigid body when it is rotating about O.

(b) *(4 Points)* Draw a free-body diagram of the rigid body when it is rotating about O. For full credit, give clear representations for the forces and moments in this diagram.

(c) *(5 Points)* Show that the following differential equation governs θ when the body is rotating about O:

$$\frac{4m\ell^2}{3} \ddot{\theta} = -mg\ell \cos(\theta) + K? \tag{7}$$

For full credit, supply the missing term.

(d) *(5 Points)* Starting from the work-energy theorem (5), prove that the total energy E of the rigid body is conserved. For full credit, supply an expression for the total energy E.

Question 2

A Rigid Body on an Incline

(20 Points)

As shown in Figure 2, a long slender rigid rod of mass m, moment of inertia relative to its center of mass C of I_{zz}, and length 2ℓ, rests with one end A on a smooth horizontal surface and the other end B on a smooth incline. The rod is supported on rigid massless rollers of negligible radii at A and B. The position vectors of the points A, C, and B, have the representations:

$$\mathbf{x}_A = x_A\mathbf{E}_x = -\frac{2\ell}{\sin(\beta)}\sin(\theta+\beta)\,\mathbf{E}_x, \qquad \bar{\mathbf{x}} = \mathbf{x}_C = \mathbf{x}_A + \ell\mathbf{e}_x,$$
$$\mathbf{x}_B = s_B\left(\cos(\beta)\mathbf{E}_x - \sin(\beta)\mathbf{E}_y\right) = -\frac{2\ell\sin(\theta)}{\sin(\beta)}\left(\cos(\beta)\mathbf{E}_x - \sin(\beta)\mathbf{E}_y\right), \tag{8}$$

where β is a constant.

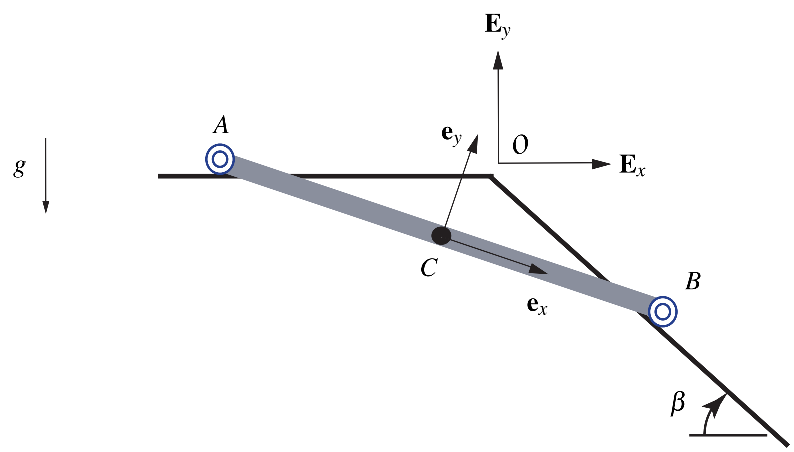

Figure 2: *A rigid body of mass m and length 2ℓ is supported at its ends by rigid massless rollers which are free to move on smooth surfaces.*

(a) *(8 Points)* Suppose the body is in motion with A in contact with the horizontal surface and B in contact with the incline. Show that the kinetic energy T and acceleration of the center of mass of the rigid body have the representations

$$T = \frac{\alpha_1}{2}\dot{\theta}^2,$$
$$\ddot{\mathbf{x}}_C = -\frac{2\ell}{\sin(\beta)}\left(\ddot{\theta}\cos(\theta+\beta) - \dot{\theta}^2\sin(\theta+\beta)\right)\mathbf{E}_x + \ell\ddot{\theta}\mathbf{e}_y - \ell\dot{\theta}^2\mathbf{e}_x, \tag{9}$$

where

$$\alpha_1 = I_{zz} + m\ell^2\left(1 + \frac{4\cos^2(\theta+\beta)}{\sin^2(\beta)} + \frac{4\cos(\theta+\beta)\sin(\theta)}{\sin(\beta)}\right). \tag{10}$$

(b) *(3 Points)* Draw a free-body diagram of the rigid body.

(c) *(4 Points)* Using the work-energy theorem, prove that the total energy E of the rigid body is conserved when it is in motion. For full credit, supply an expression for E.

(d) *(5 Points)* Using the fact that the total energy is conserved, establish the differential equation governing the motion of the rod. Your solution will be of the form

$$\alpha_1\ddot{\theta} + \alpha_2\dot{\theta}^2 + \alpha_3 = 0, \tag{11}$$

where α_1, α_2, and α_3 depend on some of the following parameters and angles: I_{zz}, m, ℓ, β, g, and θ.

Question 3

A Rolling Rigid Body

(20 Points)

As shown in Figure 3, a rigid body consists of a solid circular axle of radius r that is connected by a set of webs to a circular wheel of radius R. The combined body has a mass m and moment of inertia (relative to its center of mass C) I_{zz}. The axle rolls without slipping on a rough inclined rail. The position vector of the center of mass C has the representation

$$\bar{\mathbf{x}} = x\mathbf{E}_x + r\mathbf{E}_y. \tag{12}$$

A spring of unstretched length ℓ_0 and stiffness K is attached to the point C on the rigid body and a fixed point B. The position vector of the point B is

$$\mathbf{x}_B = r\mathbf{E}_y. \tag{13}$$

Note that in this problem x takes on negative values.

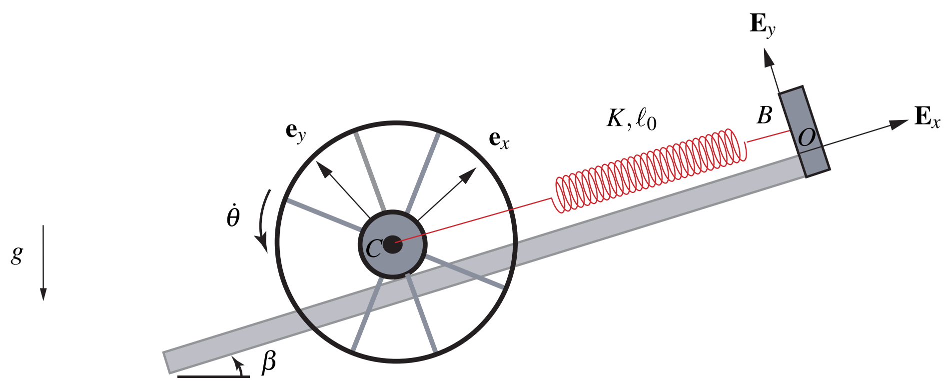

Figure 3: *A wheel of mass m with an axle of radius of r and an outer radius of R rolling on an inclined rail.*

(a) *(4 Points)* With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the rigid body, show that the slip speed v_P of the instantaneous point of contact P of the axle with the inclined rail can be expressed as

$$v_P = \dot{x} + r\dot{\theta}, \tag{14}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the rigid body.

(b) *(5 Points)* Draw a free-body diagram of the rigid body. For full credit, give a clear representation for the spring force. You will find it helpful to recall that $x < 0$ in this problem.

(c) *(3+3 Points)* Assume that the rigid body is rolling. Using a balance of linear momentum, show that

$$\mathbf{F}_f + \mathbf{N} = m\left(?? + g\left(\cos(\beta)\mathbf{E}_y + \sin(\beta)\mathbf{E}_x\right)\right) + ??? \tag{15}$$

Show that the equation governing the motion of the rolling body can be expressed as

$$\left(I_{zz} + ????\right)\ddot{\theta} = mg????? + K?????? \tag{16}$$

For full credit, supply the missing terms in (15) and (16).

(d) *(5 Points)* Suppose the plane is horizontal ($\beta = 0$) and the rigid body is released from rest at time $t = 0$ with $\theta(0) = 0$ and $x = x_0$. Establish the range of initial values for x_0 such that the rigid body will roll initially. Your solution should show that this range becomes smaller as the ratio of the gravitational force to the spring stiffness decreases.

Question 4

A Pair of Rigid Bodies *(20 Points)*

As shown in Figure 4, a uniform thin rod of mass m_1, moment of inertia about O of $I_{O_{zz}}$, and length ℓ is free to rotate about a fixed point O. At the end of the rod, a rod of mass m_2, length $2R$ and moment of inertia $I_{zz} = \frac{1}{3}m_2R^2$ about its center of mass C_2 is attached by a pin joint and is free to rotate about $\mathbf{E}_z$.

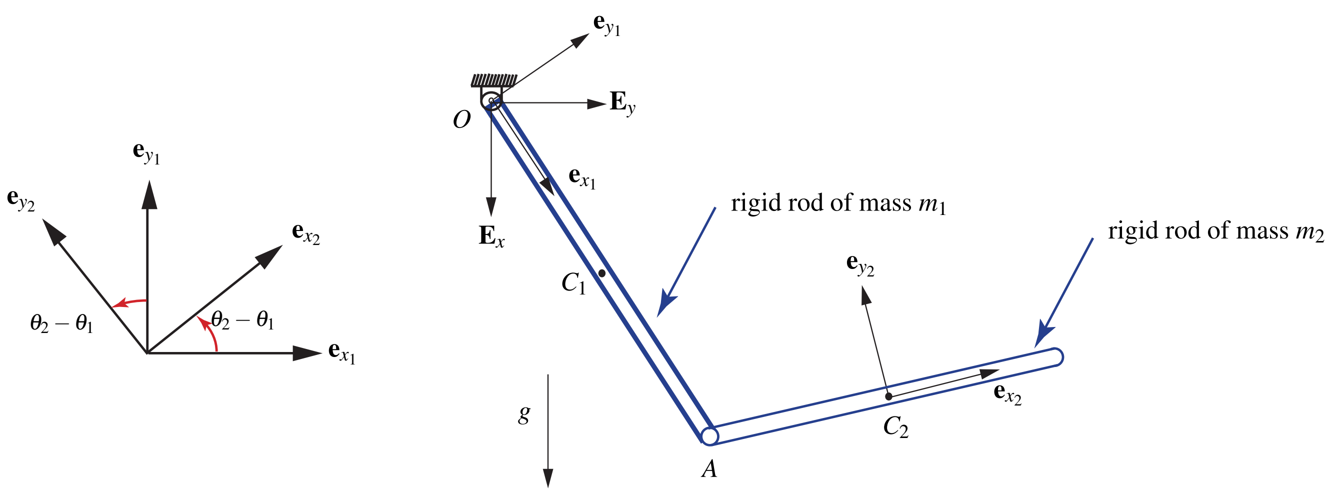

Figure 4: *A uniform rod of length ℓ and mass m_1 is free to rotate about a fixed point O. At the other end of the rod, a rod of mass m_2 and length $2R$ is free to rotate about the $\mathbf{E}_z$ axis. The sketch of the corotational bases on the left facilitates computing their cross products and inner products.*

Relative to a fixed origin O, the center of mass C_1 of the rod of length ℓ and the point C_2 have the following position vectors:

$$\bar{\mathbf{x}}_1 = \frac{\ell}{2}\mathbf{e}_{x_1}, \qquad \bar{\mathbf{x}}_2 = \ell\mathbf{e}_{x_1} + R\mathbf{e}_{x_2}, \tag{17}$$

where

$$\mathbf{e}_{x_\alpha} = \cos(\theta_\alpha)\mathbf{E}_x + \sin(\theta_\alpha)\mathbf{E}_y, \qquad \mathbf{e}_{y_\alpha} = -\sin(\theta_\alpha)\mathbf{E}_x + \cos(\theta_\alpha)\mathbf{E}_y, \qquad \alpha = 1,2. \tag{18}$$

The angular momentum of the rod of length $2R$ relative to its center of mass C_2 is

$$\mathbf{H}_{\text{rod}_2} = \frac{1}{3}m_2R^2\dot{\theta}_2\mathbf{E}_z, \tag{19}$$

where $\dot{\theta}_2\mathbf{E}_z$ is the angular velocity of the rod of length $2R$.

(a) *(5 Points)* Show that the linear momentum $\mathbf{G}$ of the system has the representation

$$\mathbf{G} = \left(m_1\frac{\ell}{2} + m_2\ell\right)\dot{\theta}_1\mathbf{e}_{y_1} + m_2R\dot{\theta}_2\mathbf{e}_{y_2}. \tag{20}$$

(b) *(7 Points)* Show that the angular momentum $\mathbf{H}_O$ of the system relative to O is

$$\mathbf{H}_O = \left(I_{O_{zz}} + m_2\ell^2\right)\dot{\theta}_1\mathbf{E}_z + \frac{4}{3}m_2R^2\dot{\theta}_2\mathbf{E}_z + ??\dot{\theta}_1\mathbf{E}_z + ???\dot{\theta}_2\mathbf{E}_z. \tag{21}$$

For full credit supply the missing terms.

(c) *(8 Points)* Show that the kinetic energy T of the system has the representation

$$T = a_1\dot{\theta}_1^2 + a_2\dot{\theta}_2^2 + a_3\dot{\theta}_1\dot{\theta}_2. \tag{22}$$

For full credit, supply expressions for the coefficients a_1, a_2, and a_3. These coefficients will depend on the parameters of the system and may also depend on the angles θ_1 and θ_2.

Question 5

A Collar on a Rotating Rod *(20 Points)*

As shown in Figure 5, a uniform thin rod of mass m_1, moment of inertia about O of $I_{O_{zz}}$, and length ℓ is free to rotate about a fixed point O. A collar of mass m_2 is attached to the end of the rod by a spring of unstretched length ℓ_0 and stiffness K. Vertical gravitational forces in the $\mathbf{E}_z$ direction act on the system and an applied moment $M_a\mathbf{E}_z$ acts on the rod.

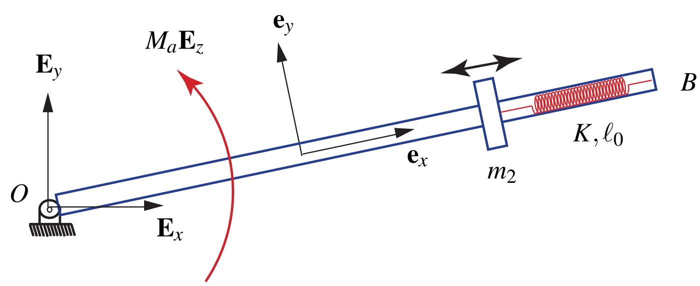

Figure 5: *A uniform rod of length ℓ and mass m_1 is free to rotate about a fixed point O and a collar of mass m_2 is attached by a spring to a point B at the end of the rod. The collar is free to move on the smooth rod. Vertical gravitational forces $-m_1g\mathbf{E}_z$ and $-m_2g\mathbf{E}_z$ act on system.*

Relative to a fixed origin O, the center of mass C of the rod of length ℓ and the collar have the following position vectors:

$$\bar{\mathbf{x}} = \frac{\ell}{2}\mathbf{e}_x, \qquad \mathbf{r} = r\mathbf{e}_x. \tag{23}$$

(a) *(5 Points)* Show that the linear momentum $\mathbf{G}$ of the system has the representation

$$\mathbf{G} = \left(m_1\frac{\ell}{2} + m_2 r\right)\dot{\theta}\mathbf{e}_y + m_2\dot{r}\mathbf{e}_x. \tag{24}$$

Show that the angular momentum of the system relative to O is

$$\mathbf{H}_O = \left(I_{O_{zz}} + m_2 r^2\right)\dot{\theta}\mathbf{E}_z. \tag{25}$$

(b) *(5 Points)* Draw freebody diagrams of (i) the collar of mass m_2, and (ii) the collar-rod system.

(c) *(5 Points)* Show that the motion of the collar is governed by the differential equation

$$m_2\left(\ddot{r} - r\dot{\theta}^2\right) + K? = 0. \tag{26}$$

For full credit supply the missing term.

(d) *(5 Points)* Show that the angle of rotation θ is governed by the differential equation

$$b_1\ddot{\theta} + b_2\dot{\theta}\dot{r} + b_3 = 0. \tag{27}$$

For full credit, supply expressions for the coefficients b_1, b_2, and b_3 in terms of the parameters $I_{O_{zz}}$, m_2, applied moment M_a, and displacement r.

Question 6

A Block Colliding with a Fixed Point (20 Points)

As shown in Figure 6, a uniform rigid block of mass m, height h, width w and moment of inertia I_{zz} traveling with a velocity $v_0\mathbf{E}_y$ and rotating with an angular velocity $\boldsymbol{\omega} = \omega_0\mathbf{E}_z$ collides with an obstacle at O. After the impact, the rigid body rotates about one of its corner points that remains in contact with O.

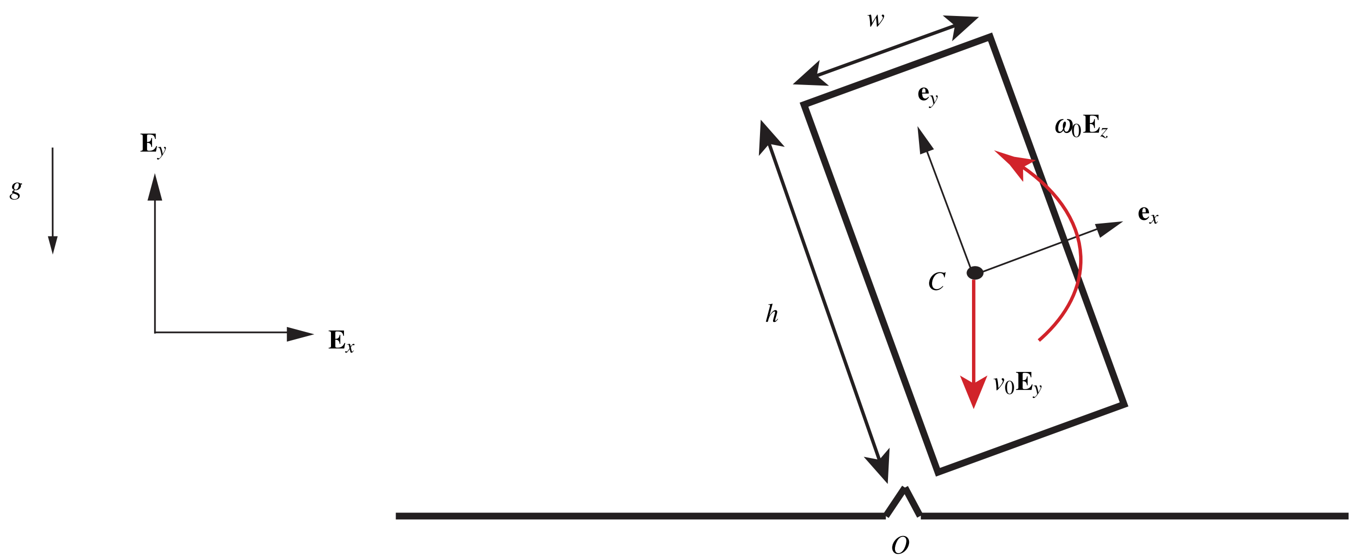

Figure 6: *A rigid body of mass m collides with a rigid obstacle at O with* $v_0 < 0$ *and* $\omega_0 > 0$. *After the impact, the rigid body is assumed to rotate about O.*

(a) *(5 Points)* Using the following representation for the position vector of the center of mass C relative to O at the instant just prior to the impact,

$$\bar{\mathbf{x}} - \mathbf{x}_O = \frac{1}{2}\left(w\left(\cos\left(\theta_0\right)\mathbf{E}_x + \sin\left(\theta_0\right)\mathbf{E}_y\right) + h\left(\cos\left(\theta_0\right)\mathbf{E}_y - \sin\left(\theta_0\right)\mathbf{E}_x\right)\right), \tag{28}$$

establish expressions for the angular momentum $\mathbf{H}_O$, kinetic energy T, and total energy E of the rigid body at the instant just before the collision.

(b) *(5 Points)* Starting from the following representation for the position vector of the center of mass C relative to O,

$$\bar{\mathbf{x}} - \mathbf{x}_O = \frac{1}{2}\left(w\mathbf{e}_x + h\mathbf{e}_y\right), \tag{29}$$

establish expressions for the angular momentum $\mathbf{H}_O$, kinetic energy T, and total energy E of the rigid body at any instant following the collision.

(c) *(5 Points)* Show that the angular velocity of the rigid body at the instant immediately following the collision is

$$\boldsymbol{\omega} = \frac{mv_0\left(w\cos\left(\theta_0\right) - h\sin\left(\theta_0\right)\right) + 2I_{zz}\omega_0}{2\left(I_{zz} + \frac{m}{4}\left(h^2 + w^2\right)\right)}\mathbf{E}_z. \tag{30}$$

(d) *(5 Points)* Suppose that $\omega_0 = 0$ (i.e., the rigid body is not rotating prior to the collision). With the help of (30), show that the energy loss due to the collision is proportional to $\frac{m}{2}v_0^2$ and that the impulse of the reaction force at O during the collision is proportional to mv_0.