Final Examination
Thursday, August 13, 2015
9:30 am to 12:00 pm
3106 Etcheverry Hall

Closed Books and Closed Notes
For Full Credit Answer All Four Questions

Useful Formulae

For all the corotational bases shown in the figures

$$\begin{aligned} \mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x. \end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{3}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{4}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

A Suspended Plate

(25 Points)

As shown in Figure 1, a thin uniform rigid plate of mass m, width $2w$ and breadth $2b$ is pinjointed at O. During the motion of the plate, a vertical gravitational force also acts on the rigid body along with a force supplied by a spring attached to a fixed point A and a point S on the plate. The position vectors of the center of mass C of the rigid body and the points S and A relative to the point O, and the angular momentum of the rigid body relative to C have the representations

$$\bar{\mathbf{x}} = b\mathbf{e}_x + w\mathbf{e}_y, \qquad \mathbf{x}_S = 2w\mathbf{e}_y, \qquad \mathbf{x}_A = 2w\mathbf{E}_y, \qquad \mathbf{H} = \left(I_{zz} = \frac{m}{3}\left(w^2 + b^2\right)\right)\dot{\theta}\mathbf{E}_z. \tag{5}$$

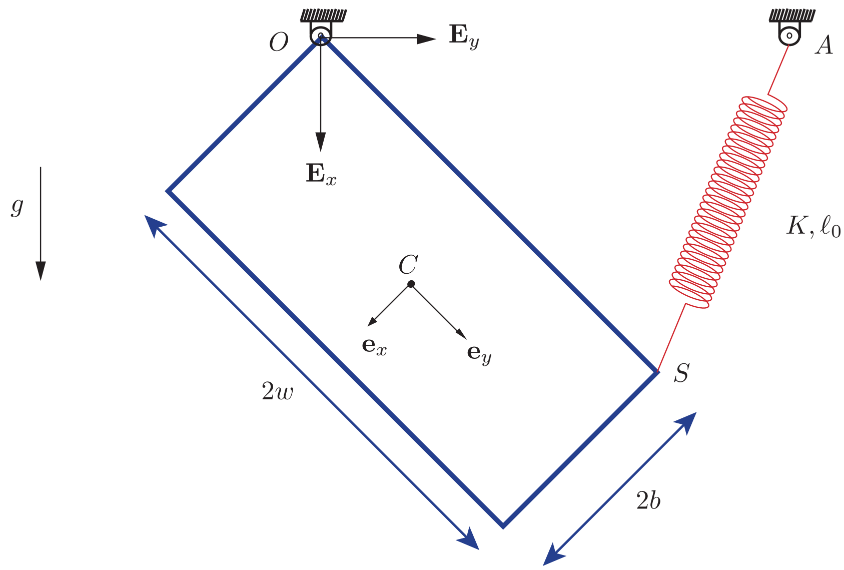

Figure 1: *A rigid body of mass m is free to rotate about O and is subject to a gravitational and spring forces.*

(a) *(6 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T of the rigid body when it is rotating about O.

(b) *(6 Points)* Draw a free–body diagram of the rigid body when it is rotating about O. Verify that the extension ϵ of the spring is $\epsilon = 2w\sqrt{2\left(1 - \cos(\theta)\right)} - \ell_0$.

(c) *(8 Points)* Show that the following differential equation governs θ when the body is rotating about O:

$$\frac{4m}{3}\left(w^2 + b^2\right)\ddot{\theta} = -mg\left(b? + w??\right) - K\frac{4w^2\epsilon}{\epsilon + \ell_0}??? \tag{6}$$

For full credit, supply the missing terms.

(d) *(5 Points)* Starting from the work-energy theorem (4), prove that the total energy E of the rigid body is conserved. For full credit, supply an expression for the total energy E.

Question 2

Rolling of a Rigid Body (25 Points)

As shown in Figure 2, a circular cylinder of radius R, mass m_1, and moment of inertia (relative to its center of mass C_1) I_{zz} is free to move atop a cart of mass m_2. A force $F(t)\mathbf{E}_x$ acts on the cart. The position vectors of the centers of mass C_1 and C_2 have the representations

$$\bar{\mathbf{x}}_1 = (x_1 + x_2)\,\mathbf{E}_x + y_0\mathbf{E}_y, \qquad \bar{\mathbf{x}}_2 = x_2\mathbf{E}_x, \tag{7}$$

where y_0 is a constant. The point P is the instantaneous point of contact of the cylinder and the cart.

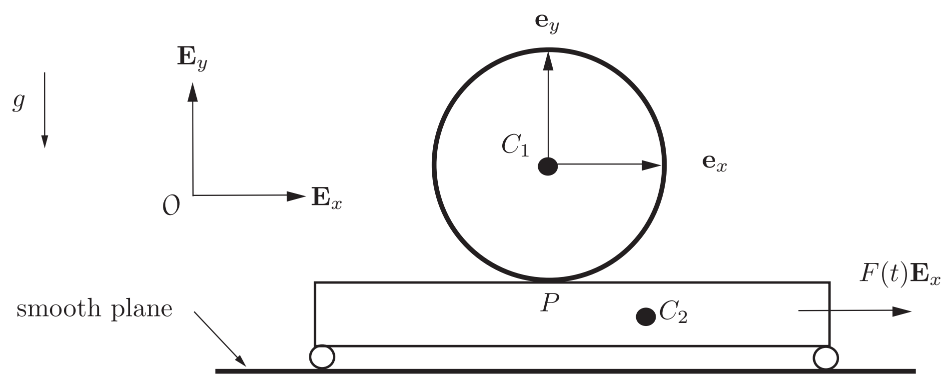

Figure 2: *A rigid cylinder of mass m and radius R moving on a cart. A force $F(t)\mathbf{E}_x$ acts on the cart.*

(a) With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the cylinder, show that rolling of the cylinder on the cart implies that

$$0 = \dot{x}_1 + R\dot{\theta}, \tag{8}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the semicylinder.

(b) Draw free-body diagrams of the cylinder and the cart. Assume that the cylinder is rolling on the cart.

(c) Assume that the cylinder is rolling. Using balances of linear momentum, show that

$$\mathbf{F}_f + \mathbf{N}_1 = m_1\,(\ddot{x}_1 + \ddot{x}_2)\,\mathbf{E}_x + m_1 g\mathbf{E}_y, \qquad (m_1 + m_2)\,\ddot{x}_2 + m_1\ddot{x}_1 = F, \tag{9}$$

where $\mathbf{F}_f$ and $\mathbf{N}_1$ are the respective friction force and normal force on the cylinder.

(d) Show that the equation of motion for the cart can be expressed as

$$(m_2 + \gamma m_1)\,\ddot{x}_2 = F, \qquad \text{where } \gamma = \frac{I_{zz}}{I_{zz} + m_1 R^2}. \tag{10}$$

(e) Show that in order for the cylinder to roll on the cart, the force F must be bounded:

$$|F|\left(\frac{\gamma m_1}{m_2 + \gamma m_1}\right) \leq \mu_s m_1 g, \tag{11}$$

where μ_s is the coefficient of static friction between the cart and the cylinder.

Question 3

Shutting the Door of a Car (25 Points)

As shown in Figure 3, a simple model to investigate the dynamics of the door of a car models the door as a rigid body of mass m and moment of inertia I_{zz} relative to its center of mass C that is attached by a pin joint at A to a collar. The motion of the collar is prescribed so as to mimic accelerating and decelerating cars and the behavior of the door is examined.

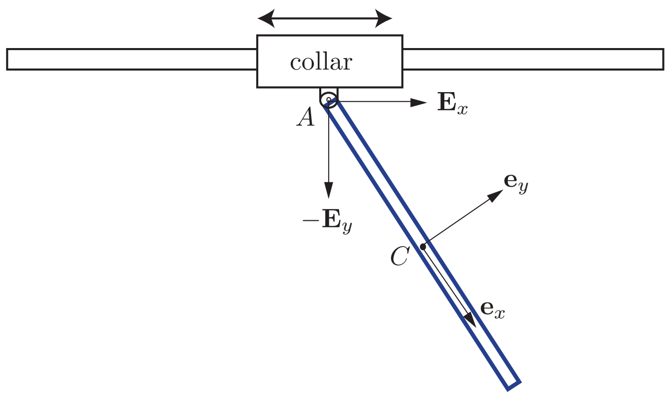

Figure 3: *A rigid body of mass m that is pinjointed to a collar that moves on a smooth track. Note that the gravitational force in this system is in the $-\mathbf{E}_z$ direction.*

(a) Starting from the following representation for the position vector of the center of mass C relative to A,

$$\bar{\mathbf{x}} - \mathbf{x}_A = \frac{\ell}{2}\mathbf{e}_x, \qquad \text{where } \mathbf{x}_A = x_A \mathbf{E}_x \tag{12}$$

establish expressions for the angular momentum $\mathbf{H}_A$ and kinetic energy T of the rigid body.

(b) Draw a freebody diagram of the rigid body.

(c) Show that the reaction force and reaction moment at A are

$$\mathbf{R}_A = mg\mathbf{E}_z + m\ddot{x}_A\mathbf{E}_x + \frac{m\ell}{2}\left(\ddot{\theta}\mathbf{e}_y - \dot{\theta}^2\mathbf{e}_x\right), \quad \mathbf{M}_c = -\frac{mg\ell}{2}\mathbf{e}_y. \tag{13}$$

(d) Show that the equation of motion for the rigid body is

$$(I_{zz} + ?)\,\ddot{\theta} = \frac{m\ell}{2}\ddot{x}_A ??. \tag{14}$$

For full credit supply the missing terms.

(e) Suppose that $\ddot{x}_A = a$ where a is a constant. Show that the following kinematical quantity

$$\frac{1}{2}\left(I_{zz} + \frac{m\ell^2}{4}\right)\dot{\theta}^2 + \frac{m\ell}{2}a\cos(\theta) \tag{15}$$

is conserved during the motion of the rigid body.

Question 4

A Vibration Demonstration *(25 Points)*

As shown in Figure 4, a pair of identical uniform thin rods of mass m and length 2ℓ are pin-jointed at their end points and free to move in a vertical plane. A spring of unstretched length $\ell_0 = 4\ell$ and stiffness K connects the rods at their other end points.

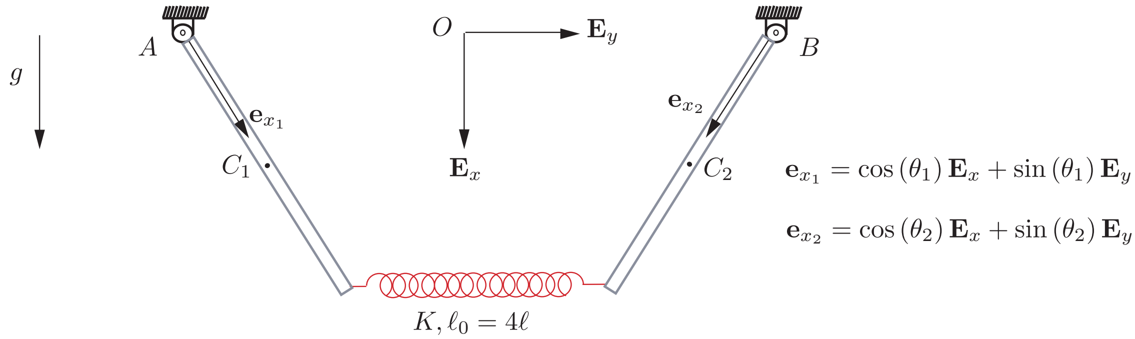

Figure 4: *A pair of pendula connected by a linear spring of stiffness K and unstretched length $\ell_0 = 4\ell$. The pendula are free to move on a smooth vertical plane.*

Relative to a fixed origin O, the centers of mass C_1 and C_2 of the rods and the fixation points A and B have the following position vectors:

$$\bar{\mathbf{x}}_1 = -2\ell\mathbf{E}_y + \ell\mathbf{e}_{x_1}, \qquad \mathbf{r}_A = -2\ell\mathbf{E}_y, \qquad \mathbf{r}_B = 2\ell\mathbf{E}_y, \qquad \bar{\mathbf{x}}_2 = 2\ell\mathbf{E}_y + \ell\mathbf{e}_{x_2}, \tag{16}$$

where $\{\mathbf{e}_{x_1}, \mathbf{e}_{y_1}, \mathbf{E}_z\}$ is a basis that corotates with the first rod and $\{\mathbf{e}_{x_2}, \mathbf{e}_{y_2}, \mathbf{E}_z\}$ is a basis that corotates with the second rod. The angular momentum of the rods relative to their respective centers of mass are

$$\mathbf{H}_1 = I_{zz}\dot{\theta}_1\mathbf{E}_z, \qquad \mathbf{H}_2 = I_{zz}\dot{\theta}_2\mathbf{E}_z, \qquad \text{where } I_{zz} = \frac{m\ell^2}{3}. \tag{17}$$

(a) *(5 Points)* Show that the kinetic energy T of the system has the representation

$$T = \frac{1}{2}I^A_{zz}\dot{\theta}_1^2 + \frac{1}{2}I^B_{zz}\dot{\theta}_2^2, \qquad \text{where } I^A_{zz} = I^B_{zz} = \frac{4m\ell^2}{3}. \tag{18}$$

(b) *(5 Points)* Draw free-body diagrams of each rod. In your solution give clear expressions for the spring forces, the extension ϵ of the spring and the potential energy U of the system.

(c) *(8 Points)* With the help of a pair of balances of angular momenta, show that the equations of motion for the system have the form

$$I^A_{zz}\ddot{\theta}_1 = -mg\ell\sin(\theta_1) + ?, \qquad I^B_{zz}\ddot{\theta}_2 = -mg\ell\sin(\theta_2) + ?? \tag{19}$$

For full credit, supply the pair of missing terms.

(d) *(5 Points)* Prove that the total energy E of the system is constant.

(e) *(2 Points)* Starting from (19), show that if the system is initially at rest with $\theta_1 = \theta_2 = 0$, then it will remain at rest.