Final Examination
Tuesday May 12, 2015
7:00pm to 10:00 pm
F295 Haas

Closed Books and Closed Notes
For Full Credit Answer All Four Questions

Useful Formulae

For the corotational bases shown in the figures:

$$\begin{aligned}\mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x.\end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}, \tag{3}$$

We also note that given an vector $\mathbf{a} = \mathbf{a}(t)$ then

$$\frac{d\,\|\mathbf{a}\|}{dt} = \frac{\mathbf{a} \cdot \dot{\mathbf{a}}}{\|\mathbf{a}\|}. \tag{4}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{5}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1
Motion of a Rigid Rod
(25 Points)

As shown in Figure 1, a thin uniform rod of mass m and length 2ℓ is pin-jointed at O. A force $\mathbf{P}$ of constant magnitude P is transmitted by a cable to point A which is located at the end of the rod. During the ensuing motion, a vertical gravitational force $mg\mathbf{E}_x$ also acts on the rigid body.

The position vectors of the center of mass C of the rigid body, the point A relative to the point O, and the point B relative to O, and the angular momentum of the rigid body relative to C have the representations

$$\bar{\mathbf{x}} = \ell \mathbf{e}_x, \qquad \mathbf{x}_A = 2\ell \mathbf{e}_x, \qquad \mathbf{x}_B = 2\ell \mathbf{E}_y, \qquad \mathbf{H} = \left(I_{zz} = \frac{m\ell^2}{3} \right) \dot{\theta} \mathbf{E}_z. \tag{6}$$

We also note that

$$\| \mathbf{x}_B - \mathbf{x}_A \| = 2\ell \sqrt{2\left(1 - \sin(\theta)\right)}. \tag{7}$$

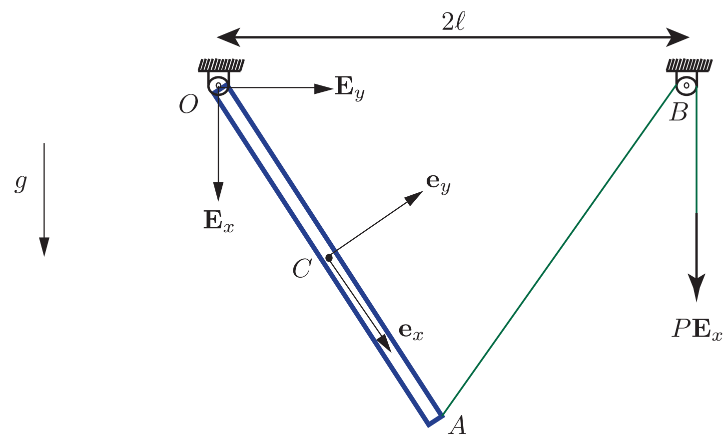

Figure 1: *A rigid body of mass m is free to rotate about O. The rigid body is subject to a gravitational force and a force $\mathbf{P}$ which has a constant magnitude.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T of the rigid body when it is rotating about O.

(b) *(5 Points)* Draw a free-body diagram of the rigid body when it is rotating about O. For credit, give clear representations for the forces and moments in this diagram.

(c) *(5 Points)* Show that the following differential equation governs θ when the body is rotating about O:

$$\frac{4m\ell^2}{3}\ddot{\theta} = -mg\ell \sin(\theta) + \frac{2P\ell \cos(\theta)}{\sqrt{2(1 - \sin(\theta))}}. \tag{8}$$

(d) *(5 Points)* Show that the force $\mathbf{P}$ has a potential energy function

$$U_P = P \| \mathbf{x}_B - \mathbf{x}_A \|. \tag{9}$$

(e) *(5 Points)* Starting from the work-energy theorem (5), prove that the total energy E of the rigid body is conserved. For credit, supply an expression for the total energy E.

Question 2

A Falling Ladder (25 Points)

As shown in Figure 2, a ladder of mass m, moment of inertia relative to its center of mass C of I_{zz} and length 2ℓ rests with one end A on a smooth horizontal surface and the other end B on a smooth vertical wall. The position vectors of these points have the representations:

$$\mathbf{x}_A = x_A\mathbf{E}_x = -2\ell\cos(\theta)\mathbf{E}_x, \qquad \mathbf{x}_B = y_A\mathbf{E}_y = 2\ell\sin(\theta)\mathbf{E}_y, \qquad \bar{\mathbf{x}} = \mathbf{x}_A + \ell\mathbf{e}_x. \tag{10}$$

Initially, a stop is placed at A (see Figure 2(a)) to prevent the ladder from falling. At a later time, the stop is removed and the ladder falls (see Figure 2(b)).

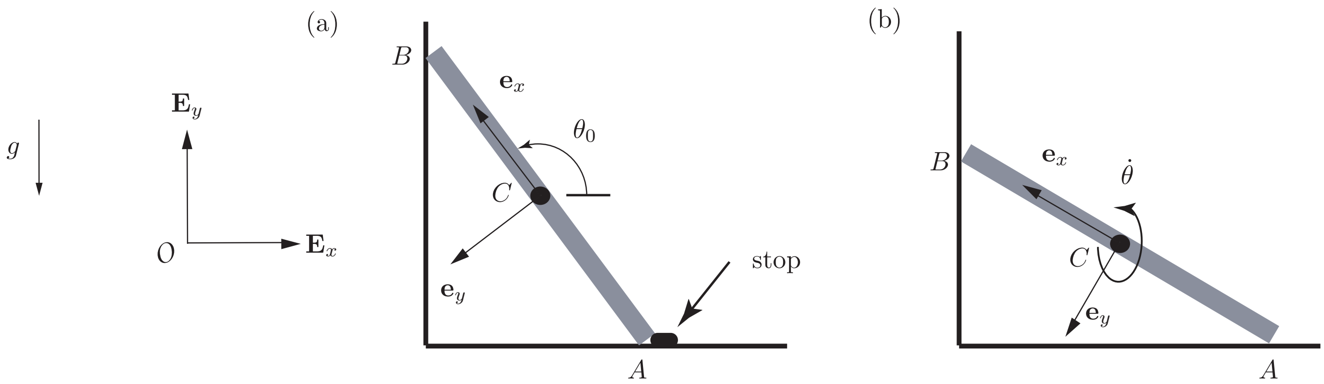

Figure 2: *(a) A ladder of mass m rests at an inclination angle $\theta = \theta_0$ with the help of a stop at A. (b) The stop is removed and the ladder falls.*

(a) *(5 Points)* Suppose the ladder is in motion with A in contact with the ground and B in contact with the wall. Show that the acceleration of the center of mass of the ladder has the representation

$$\ddot{\mathbf{x}}_C = 2\ell\left(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\right)\mathbf{E}_x + \ell\ddot{\theta}\mathbf{e}_y - \ell\dot{\theta}^2\mathbf{e}_x. \tag{11}$$

(b) *(3 Points)* Draw a free-body diagram of the ladder. Distinguish the cases where the ladder is stationary and when it is sliding.

(c) *(5 Points)* Assume that the ladder is stationary, inclined an angle $\theta = \theta_0$ and the stop at A is in place. Show that the forces acting on the ladder at A and B are

$$\mathbf{R}_A = \frac{mg}{2}\cot(\theta_0)\mathbf{E}_x + mg\mathbf{E}_y, \qquad \mathbf{R}_B = -\frac{mg}{2}\cot(\theta_0)\mathbf{E}_x. \tag{12}$$

(d) *(7 Points)* The stop at A is removed and the ladder starts to move. Show that the differential equation governing the motion of the ladder is

$$(I_{zz} + ?)\,\ddot{\theta} = -??\ell\cos(\theta). \tag{13}$$

For credit, supply the missing terms.

(e) *(5 Points)* Using the work-energy theorem, prove that the total energy E of the ladder is conserved when it is in motion. Show that the angular velocity ω of the ladder at the instant before it becomes horizontal is

$$\omega = \sqrt{\frac{2??\ell\sin(\theta_0)}{(I_{zz} + m\ell^2)}}. \tag{14}$$

For full credit provide the missing term in the above equation and give an expression for E.

Question 3

A Rolling Rigid Body (25 Points)

As shown in Figure 3, a rigid body consists of a solid circular cylinder of radius r that is welded to the inner surface of a hollow circular cylinder of radius R. The combined body has a mass m and moment of inertia (relative to its center of mass C) I_{zz} and is free to move on an inclined plane. The position vector of the geometric center A and the center of mass C have the representations

$$\mathbf{x}_A = x\mathbf{E}_x + y_0\mathbf{E}_y, \qquad \bar{\mathbf{x}} = \mathbf{x}_A + h\mathbf{e}_y, \tag{15}$$

where y_0 and h are constants. The point P is the instantaneous point of contact of the body with the horizontal plane.

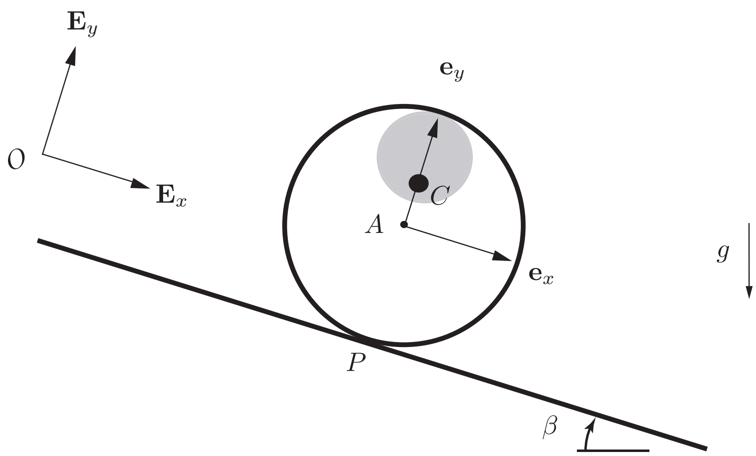

Figure 3: *A rigid body of mass m moving on a rough inclined plane.*

(a) *(5 Points)* With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the rigid body, show that the slip speed v_P of the point P can be expressed as

$$v_P = \dot{x} + R\dot{\theta}, \tag{16}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the rigid body.

(b) *(3 Points)* Draw a free-body diagram of the rigid body. Distinguish the cases where the body is rolling and where it is sliding. The coefficients of static and dynamic friction are μ_s and μ_k, respectively.

(c) *(5+7 Points)* Assume that the rigid body is rolling. Using a balance of linear momentum, show that

$$\mathbf{F}_f + \mathbf{N} = m\left(??+???+???? + g\left(\cos(\beta)\mathbf{E}_y - \sin(\beta)\mathbf{E}_x\right)\right). \tag{17}$$

Show that the equation governing the motion of the rolling body can be expressed as

$$\left(I_{zz}^P = I_{zz} + mR^2 + mh^2 + 2mhR\cos(\theta)\right)\ddot{\theta} = mRh\dot{\theta}^2\sin(\theta) + mgh??? + mgR????. \tag{18}$$

For full credit, supply the missing terms in (17) and (18).

(d) *(5 Points)* Suppose the rigid body is released from rest at time $t = 0$ with $\theta(0) = 0$. Assuming rolling, determine if the body will initially rotate clockwise or counterclockwise and whether the friction force initially points up or down the incline?

Question 4

Freezing up a Joint   (25 Points)

As shown in Figure 4, a uniform thin rod of mass m_1, moment of inertia about O of $I_{O_{zz}}$, and length ℓ is free to rotate about a fixed point O. At the end of the rod, a disk of mass m_2, radius R and moment of inertia $I_{zz} = \frac{1}{2}m_2R^2$ about its center of mass A is free to rotate.

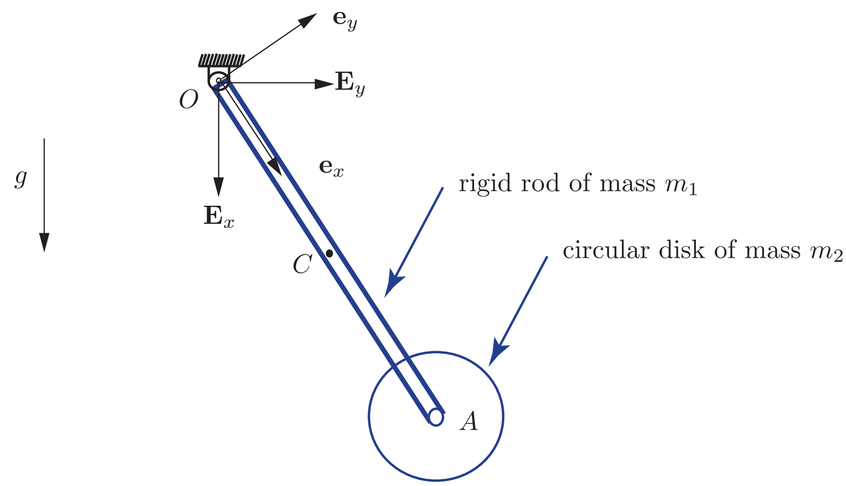

Figure 4: *A uniform rod of length ℓ and mass m_1 is free to rotate about a fixed point O. At the other end of the rod, a disk of mass m_2 and radius R is free to rotate about the $\mathbf{E}_z$ axis.*

Relative to a fixed origin O, the center of mass C of the rod and the point A have the following position vectors:

$$\bar{\mathbf{x}} = \frac{\ell}{2}\mathbf{e}_x, \qquad \mathbf{x}_A = \ell\mathbf{e}_x. \tag{19}$$

The angular momentum of the disk relative to its center of mass A is

$$\mathbf{H}_{\text{disk}} = \frac{1}{2}m_2R^2\omega\mathbf{E}_z. \tag{20}$$

where $\omega\mathbf{E}_z$ is the angular velocity of the disk. Note that $\omega \neq \dot{\theta}$.

(a) *(7 Points)* Show that the angular momentum $\mathbf{H}_O$ of the system relative to O is

$$\mathbf{H}_O = \left(I_{O_{zz}} + m_2\ell^2\right)\dot{\theta}\mathbf{E}_z + \frac{1}{2}m_2R^2\omega\mathbf{E}_z. \tag{21}$$

Establish an expression for the kinetic energy T of the system.

(b) *(3 Points)* Draw a free-body diagram of the system.

(c) *(5 Points)* Show that equations of motion for the system are

$$\frac{1}{2}m_2R^2\dot{\omega} = 0, \qquad \left(I_{O_{zz}} + m_2\ell^2\right)\ddot{\theta} = -\left(\frac{m_1\ell}{2} + m_2\ell\right)g\sin(\theta). \tag{22}$$

(d) *(5 Points)* Give an expression for the total energy E of the system and then, with the help of (22), show that $\dot{E} = 0$.

(e) *(5 Points)* Suppose the joint at A freezes when $\theta = 0$, $\dot{\theta} = 0$ and $\omega = \omega_0$. Determine the angular velocity of the system immediately following this event. OUTLINE how you would compute the minimum ω_0 needed so that the system can become horizontal (i.e., $\theta = \frac{\pi}{2}$) during the ensuing motion.