Final Examination
Thursday, August 14, 2014
9:30 am to 12:00 pm
3106 Etcheverry Hall

Closed Books and Closed Notes
For Full Credit Answer All Four Questions

Useful Formulae

For all the corotational bases shown in the figures

$$\begin{aligned} \mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x. \end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{3}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{4}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

A Hanging Plate

(25 Points)

As shown in Figure 1, a thin uniform rigid plate of mass m, width $2w$ and breadth $2b$ is pinjointed at O and A. At time $t = 0$, the pin at A is released and the plate rotates about O. During the ensuing motion, a vertical gravitational force also acts on the rigid body.

The position vectors of the center of mass C of the rigid body and the point A relative to the point O, and the angular momentum of the rigid body relative to C have the representations

$$\bar{\mathbf{x}} = w\mathbf{e}_x - b\mathbf{e}_y, \qquad \mathbf{x}_A = x_A\mathbf{E}_x, \qquad \mathbf{H} = \left(I_{zz} = \frac{m}{3}\left(w^2 + b^2\right)\right)\dot{\theta}\mathbf{E}_z. \tag{5}$$

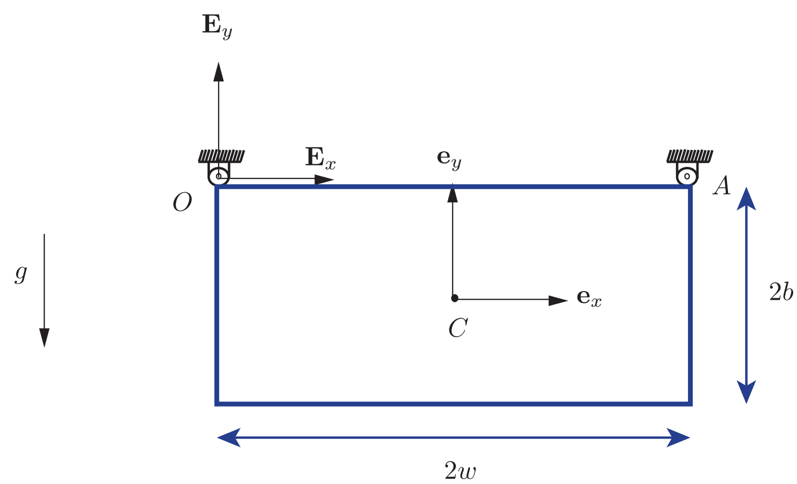

Figure 1: *A rigid body of mass m is free to rotate about O (after the pin joint at A is released) and is subject to a gravitational force.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T of the rigid body when it is rotating about O.

(b) *(5 Points)* Draw a free–body diagram of the rigid body when it is rotating about O.

(c) *(5 Points)* Show that the following differential equation governs θ when the body is rotating about O:

$$\frac{4m}{3}\left(w^2 + b^2\right)\ddot{\theta} = -mg\left(b\sin(\theta) + w\cos(\theta)\right). \tag{6}$$

(d) *(5 Points)* Starting from the work-energy theorem (4), prove that the total energy E of the rigid body is conserved. For full credit supply an expression for the total energy E.

(e) *(5 Points)* With the help of a balance of linear momentum, show that the reaction force at O during the motion of the rigid body can be expressed in terms of the energy E at time $t = 0$, m, w, b, g and θ.

Question 2

Rolling of a Rigid Body (25 Points)

As shown in Figure 2, a semicircular cylinder of radius R, mass m, and moment of inertia (relative to its center of mass C) I_{zz} is free to move on a horizontal plane. A constant force $-F_0\mathbf{E}_y$ acts at a point B on the rigid body. The position vectors of A, B, and the center of mass C have the representations

$$\mathbf{x}_A = x\mathbf{E}_x + y_0\mathbf{E}_y, \qquad \mathbf{x}_B = \mathbf{x}_A - R\mathbf{e}_x, \qquad \bar{\mathbf{x}} = \mathbf{x}_A - h\mathbf{e}_y, \tag{7}$$

where y_0 and h are constants. The point P is the instantaneous point of contact of the semicylinder with the horizontal plane.

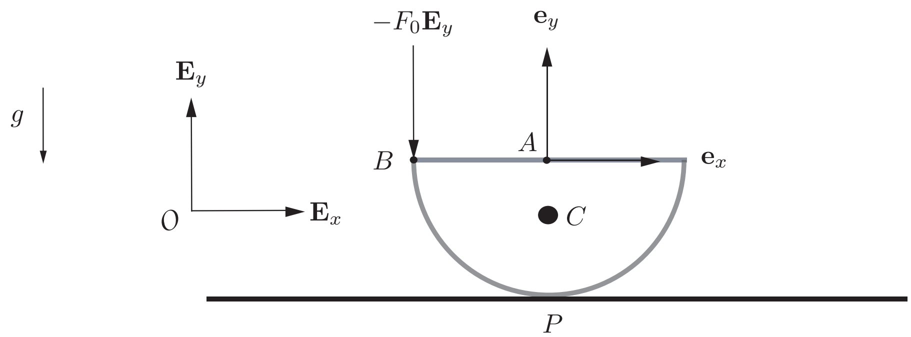

Figure 2: *A rigid semicylinder of mass m and radius R moving on a horizontal plane. A constant force $-F_0\mathbf{E}_y$ acts at a point B on the semicylinder.*

(a) *(5 Points)* With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the semicylinder, show that the slip speed v_P of the point P can be expressed as

$$v_P = \dot{x} + R\dot{\theta}. \tag{8}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the semicylinder.

(b) *(5 Points)* Show that the total energy E of the semicylinder can be expressed as a function of v_P, $\dot{\theta}$, and θ.

(c) *(3 Points)* Draw a free-body diagram of the rigid body. Distinguish the cases where the body is rolling and where it is sliding. The coefficients of static and dynamic friction are μ_s and μ_k, respectively.

(d) *(7 Points)* Assume that the rigid body is rolling. Using a balance of linear momentum, show that

$$\mathbf{F}_f + \mathbf{N} = m\left(??+???+???? + g\mathbf{E}_y\right) + F_0\mathbf{E}_y. \tag{9}$$

With the help of (9), show that the equation governing the motion of the rolling semicylinder can be expressed as

$$\left(I_{zz} + mR^2 + mh^2 - 2mhR\cos(\theta)\right)\ddot{\theta} = -mRh\dot{\theta}^2\sin(\theta) - mgh\sin(\theta) + F_0\left(?????\right). \tag{10}$$

For full credit supply the missing terms in (9) and (10).

(e) *(5 Points)* Suppose the rigid body is at rest at time $t = 0$ and $-F_0\mathbf{E}_y$ is applied where $F_0 > 0$. Determine the limiting value F_0^* of F_0 such that the rigid body will start to roll at $t = 0$.

Question 3

The Tipping Block (25 Points)

As shown in Figure 3, a uniform rigid block of mass m, height h, width w and moment of inertia I_{zz} traveling with a velocity $v_0\mathbf{E}_x$ collides with an obstacle at O. After the impact, the rigid body rotates about one of its corner points that remains in contact with O. Eventually, one of its sides collides with the ground plane.

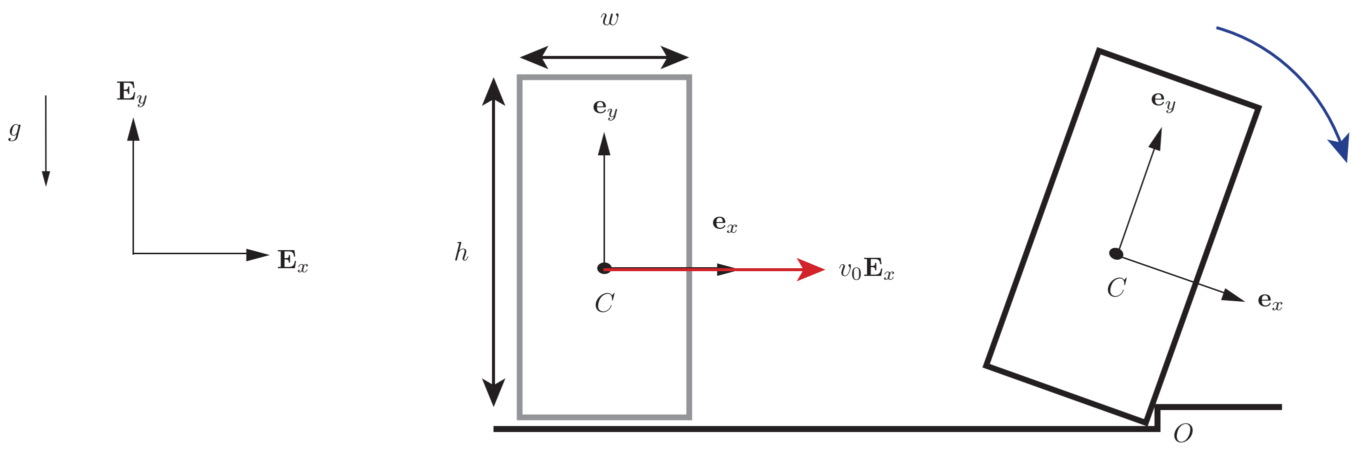

Figure 3: *A rigid body of mass m moves atop a smooth horizontal surface and collides with a ledge at O. After the impact the rigid body rotates about O.*

(a) *(7 Points)* Starting from the following representation for the position vector of the center of mass C relative to O,

$$\bar{\mathbf{x}} - \mathbf{x}_O = x\mathbf{E}_x + \frac{h}{2}\mathbf{E}_y, \tag{11}$$

establish expressions for the angular momentum $\mathbf{H}_O$, kinetic energy T, and total energy E of the rigid body at the instant just before the collision.

(b) *(7 Points)* Starting from the following representation for the position of the center of mass C relative to O,

$$\bar{\mathbf{x}} - \mathbf{x}_O = \frac{1}{2}\left(-w\mathbf{e}_x + h\mathbf{e}_y\right), \tag{12}$$

establish expressions for the angular momentum $\mathbf{H}_O$, kinetic energy T, and total energy E of the rigid body at any instant following the collision.

(c) *(7 Points)* Show that the angular velocity of the rigid body at the instant immediately following the collision is

$$\boldsymbol{\omega} = -\frac{mhv_0}{2\left(I_{zz} + \frac{m}{4}\left(h^2 + w^2\right)\right)}\mathbf{E}_z. \tag{13}$$

With the help of (13), show that the energy loss due to the collision is proportional to $\frac{m}{2}v_0^2$ and that the impulse of the reaction force at O during the collision is proportional to mv_0.

(d) *(4 Points)* Show that the equation of motion for the rigid body at any instant after the collision and prior to θ reaching $-\frac{\pi}{2}$ is

$$\left(I_{zz} + ?\right)\ddot{\theta} = -\frac{mg}{2}\left(?? + ???\right). \tag{14}$$

For full credit supply the missing terms.

Question 4

Equilibrium of a Suspended Rod *(25 Points)*

As shown in Figure 4, a uniform thin rod of mass m and length 2ℓ is suspended from its end points by two identical linear springs each of stiffness K and unstretched length ℓ_0. The bar is free to move on a smooth vertical plane.

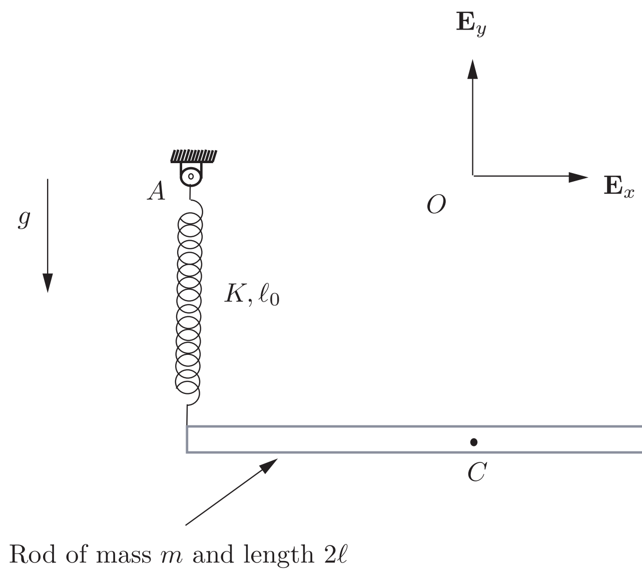

Figure 4: *A uniform rod of length 2ℓ is suspended by two identical linear springs. The rod is free to move on a smooth vertical plane.*

Relative to a fixed origin O, the center of mass C of the rod and the fixation points A and B have the following position vectors:

$$\bar{\mathbf{x}} = x\mathbf{E}_x + (y + y_0)\,\mathbf{E}_y, \qquad \mathbf{r}_A = -\ell\mathbf{E}_x, \qquad \mathbf{r}_B = \ell\mathbf{E}_x, \tag{15}$$

where ℓ and y_0 are constants. The angular momentum of the rod relative to its center of mass C is

$$\mathbf{H} = I_{zz}\dot{\theta}\mathbf{E}_z. \tag{16}$$

(a) *(5 Points)* Show that the angular momentum $\mathbf{H}_O$ of the rod relative to O has the representation

$$\mathbf{H}_O = I_{zz}\dot{\theta}\mathbf{E}_z + m\left(x\dot{y} - (y + y_0)\,\dot{x}\right)\mathbf{E}_z. \tag{17}$$

Establish an expression for the kinetic energy T of the rod.

(b) *(5 Points)* Draw a free-body diagram of the rod. In your solution give clear expressions for the spring forces.

(c) *(5 Points)* Show that equations of motion for the rod are

$$m\ddot{x} = ??+???, \qquad m\ddot{y} = -mg + ???? + ????? \qquad I_{zz}\ddot{\theta} = ?????? + ??????? \tag{18}$$

For full credit supply the six missing terms.

(d) *(5 Points)* Starting from the work-energy theorem (4), prove that the total energy E of the rod is constant. For full credit, give an expression for E.

(e) *(5 Points)* Starting from (18), show that the following condition must hold in order for the rod to be at rest with $y = x = \theta = 0$:

$$y_0 = -\frac{mg}{2K} - \ell_0. \tag{19}$$