Final Examination
Thursday, August 15, 2013
10:00 am to 12:30 pm
101 Barker Hall

Closed Books and Closed Notes
For Full Credit Answer All Four Questions

Useful Formulae

For all the corotational bases shown in the figures

$$\begin{aligned} \mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x. \end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{3}$$

The following derivative is also recorded:

$$\frac{d}{dy}\left(\frac{ay}{b + cy^2}\right) = \frac{a\left(b - cy^2\right)}{\left(b + cy^2\right)^2} \tag{4}$$

where a, b, and c are constant.

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{5}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

A Moving Van (25 Points)

As shown in Figure 1, a rigid body of mass m is attached by a pin-joint at A to the flat bed of a truck. The rotation of the body about A is resisted by a torsional spring of stiffness K. The spring exerts a restoring moment $\mathbf{M}_s$ on the rigid body:

$$\mathbf{M}_s = -K\theta\mathbf{E}_z. \tag{6}$$

A vertical gravitational force also acts on the rigid body. The position vectors of the center of mass C of the rigid body and the point A, and the angular momentum of the rigid body relative to its center of mass have the representations

$$\bar{\mathbf{x}} = \mathbf{x}_A + \ell\mathbf{e}_y, \qquad \mathbf{x}_A = x_A\mathbf{E}_x, \qquad \mathbf{H} = I_{zz}\dot{\theta}\mathbf{E}_z. \tag{7}$$

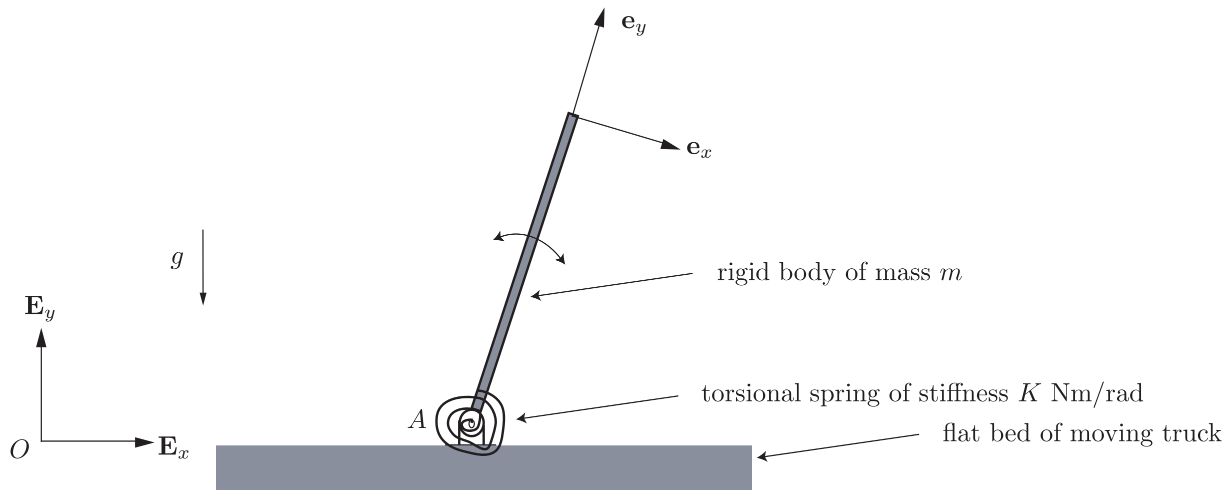

Figure 1: *A rigid body of mass m is free to rotate about A and is subject to a gravitational force and a torsional spring.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_A$ and kinetic energy T of the rigid body.

(b) *(5 Points)* Draw a free–body diagram of the rigid body.

(c) *(8 Points)* Using balances of linear and angular momentum, show that

$$\left(I_{zz} + m\ell^2\right)\ddot{\theta} + K\theta = m\ell\left(g\sin(\theta) + \ddot{x}_A\cos(\theta)\right). \tag{8}$$

(d) *(5 Points)* Starting from the work-energy theorem (5), prove that the total energy E of the rigid body changes according to

$$\dot{E} = m\ddot{x}_A\dot{x}_A + ?. \tag{9}$$

For full credit supply an expression for the total energy E and the missing term.

(e) *(2 Points)* Suppose the truck is moving with constant speed $v_0\mathbf{E}_x$ ($v_0 > 0$) and the rigid body is initially at rest with $\theta = 0°$. In which direction will the rigid body tend to rotate if the truck suddenly brakes?

Question 2

Rolling of a Rigid Body (25 Points)

As shown in Figure 2, a semicircular cylinder of radius R, mass m, and moment of inertia (relative to its center of mass) I_{zz} rolls on a cart. The position vectors of A and the center of mass C have the representations

$$\mathbf{x}_A = x\mathbf{E}_x + y_0\mathbf{E}_y, \qquad \bar{\mathbf{x}} = \mathbf{x}_A - h\mathbf{e}_y, \tag{10}$$

where y_0 and h are constants. The point P is the instantaneous point of contact of the semicylinder with the cart. The cart is in motion with respective velocity and acceleration vectors

$$\mathbf{v}_{\text{cart}} = v_c\mathbf{E}_x, \qquad \mathbf{a}_{\text{cart}} = a_c\mathbf{E}_x. \tag{11}$$

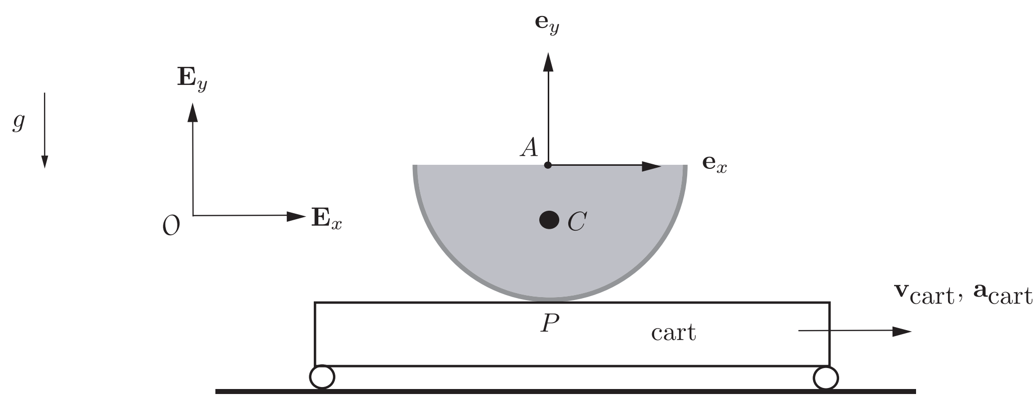

Figure 2: *A rigid semicylinder of mass m and radius R rolling on a moving cart.*

(a) *(5 Points)* With the help of the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the semicylinder, show that

$$\dot{x} + R\dot{\theta} = v_c \tag{12}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the semicylinder.

(b) *(5 Points)* Show that the total energy E of the semicylinder can be expressed as a function of v_c, $\dot{\theta}$, and θ.

(c) *(3 Points)* Draw a free-body diagram of the rolling rigid body.

(d) *(8 Points)* Using a balance of linear momentum, show that

$$\mathbf{F}_f + \mathbf{N} = m\left(a_c\mathbf{E}_x + ?? + ??? + ???? + g\mathbf{E}_y\right). \tag{13}$$

With the help of (13), show that the equation governing the motion of the semicylinder can be expressed as

$$\left(I_{zz} + mR^2 + mh^2 - 2mhR\cos(\theta)\right)\ddot{\theta} = -mRh\dot{\theta}^2\sin(\theta) - mgh\sin(\theta) + ma_c\,(?????)\,. \tag{14}$$

For full credit supply the missing terms in (13) and (14).

(e) *(4 Points)* Starting from the work-energy theorem (5), prove that the total energy E of the rolling rigid body is not typically conserved.

Question 3

Designing an Impacter *(25 Points)*

As shown in Figure 3, an impacter in the form of a rigid body of mass m and moment of inertia I_{zz} is being designed. The impacter pivots from a pin-joint at point O. After being released from rest in the horizontal position, the point S on the rigid body impacts the intended target when the body is vertical.

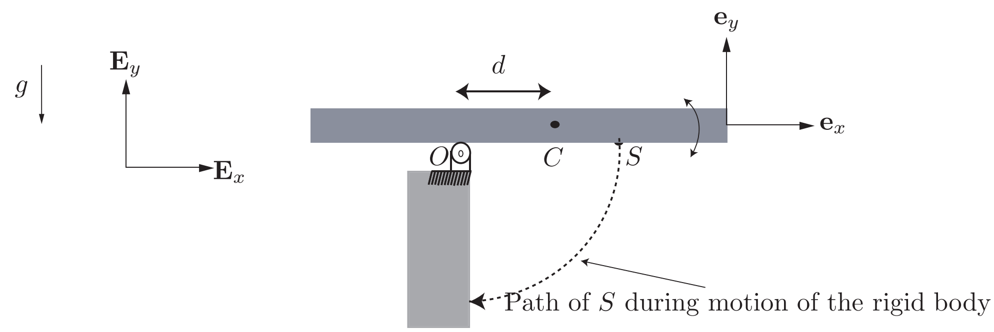

Figure 3: *A rigid body of mass m is pinpointed at a fixed point O. The body is released from rest and the point S impacts with the rigid object show when the body's inclination is vertical.*

(a) *(7 Points)* Starting from the following representation for the position of the center of mass C relative to O,

$$\bar{\mathbf{x}} = d\mathbf{e}_x, \tag{15}$$

establish expressions for the angular momentum $\mathbf{H}_O$, kinetic energy T, and total energy E of the rigid body.

(b) *(3 Points)* Draw a free body diagram of the rigid body during the segment where it is in motion prior to the impact.

(c) *(5 Points)* Show that the equation of motion for the rigid body is

$$\left(I_{zz} + md^2\right)\ddot{\theta} = -mgd\cos(\theta). \tag{16}$$

(d) *(5 Points)* If the body is released from rest at $\theta = 0$, show that the velocity vector of the point S on the body immediately prior to impact, which we denote by $\mathbf{v}_S^*$, is

$$\mathbf{v}_S^* = -\ell\sqrt{\frac{2mgd}{I_{zz} + md^2}}\mathbf{E}_x. \tag{17}$$

Here, ℓ is the distance that the point S is from O on the rigid body.

(e) *(5 Points)* How should d be chosen so as to maximize the impact velocity $\|\mathbf{v}_S^*\|$ and $\left|\ddot{\theta}\right|$? Give a clear justification for your answer.

Question 4

A Swinging Bar (25 Points)

As shown in Figure 4, a uniform thin rod of mass m and length 2ℓ is attached at one of its extremities to a fixed point O by a linear spring of stiffness K and unstretched length L_0. The bar is free to move on a smooth vertical plane.

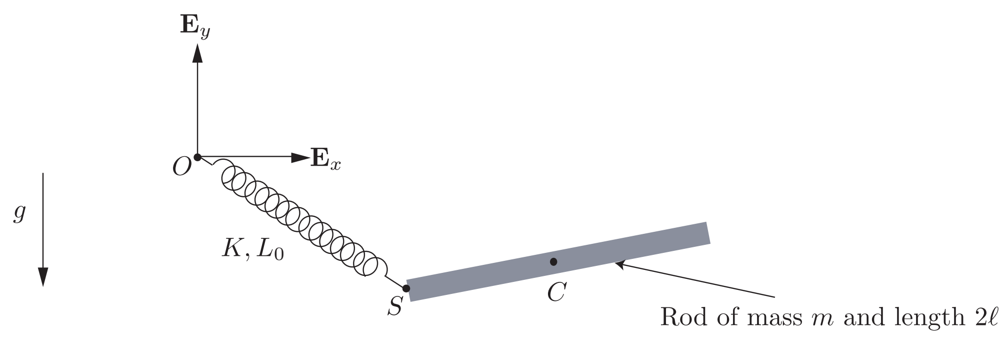

Figure 4: *A uniform rod of length 2ℓ is attached to a fixed point O by a linear spring of stiffness K and unstretched length L_0 . The rod is free to move on a smooth vertical plane.*

The center of mass C of the rod has a position vector

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y\mathbf{E}_y. \tag{18}$$

The angular momentum of the rod relative to its center of mass C is

$$\mathbf{H} = I_{zz}\dot{\theta}\mathbf{E}_z. \tag{19}$$

(a) *(5 Points)* Show that the angular momentum $\mathbf{H}_O$ of the rod relative to O has the representation

$$\mathbf{H}_O = I_{zz}\dot{\theta}\mathbf{E}_z + m\left(x\dot{y} - y\dot{x}\right)\mathbf{E}_z. \tag{20}$$

Establish an expression for the kinetic energy T of the rod.

(b) *(5 Points)* Draw a free-body diagram of the rod. In your solution give a clear expression for the spring force.

(c) *(5 Points)* Show that equations of motion for the rod are

$$m\ddot{x} = ??, \qquad m\ddot{y} = -mg - ??? \qquad I_{zz}\ddot{\theta} = ???? \tag{21}$$

For full credit supply the missing terms.

(d) *(5 Points)* Starting from the work-energy theorem (5), prove that the total energy E of the rod is constant. For full credit, give an expression for E.

(e) *(5 Points)* If gravity is ignored, which angular momentum of the rod is conserved? Clearly justify your answer.

Question 1

A Moving Van (25 Points)

Solution to Question 2

Rolling of a Rigid Body (25 Points)

Question 3

Designing an Impacter (25 Points)

Question 4

A Swinging Bar (25 Points)