Final Examination
Monday December 10, 2012
8:00 am to 11:00 am
234 Hearst Gym

Closed Books and Closed Notes
For Full Credit Answer All Four Questions

Useful Formulae

For all the corotational bases shown in the figures

$$\begin{aligned}\mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x.\end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{3}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{4}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

The Braking of a Rolling Rigid Body *(30 Points)*

As shown in Figure 1, a rigid cylinder of mass m and radius R is in motion on a rough incline. The moment of inertia (relative to its center of mass C) of the body is I_{zz}, and the position vector of C has the representation

$$\bar{\mathbf{x}} = x\mathbf{E}_x + h\mathbf{E}_y, \tag{5}$$

where h is a constant. The point P in Figure 1 is the instantaneous point of contact.

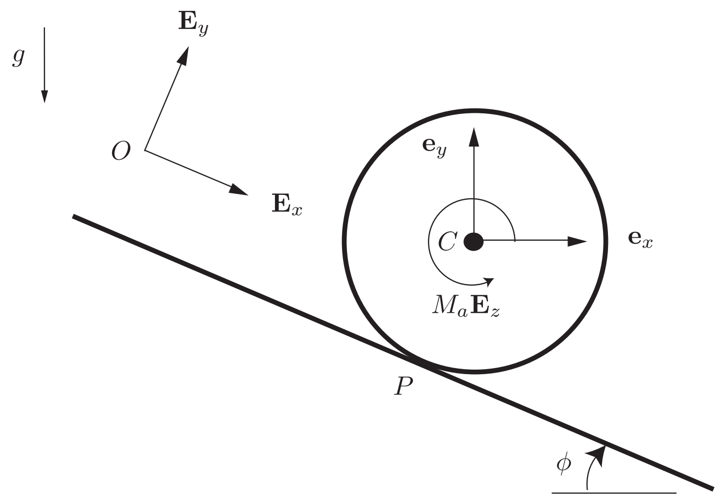

Figure 1: *A rigid body of mass m and radius R rolling under the influence of an applied torque $M_a\mathbf{E}_z$ on an inclined plane.*

(a) *(5 Points)* Using the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the rigid body, show that

$$\dot{x} + R\dot{\theta} = 0, \tag{6}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the rigid body.

(b) *(5 Points)* Draw a free-body diagram of the rolling rigid body.

(c) *(9 Points)* Using balances of linear and angular momenta, show that

$$\begin{aligned} \mathbf{F}_f &= -\left(? + mg\sin(\phi)\right)\mathbf{E}_x, \\ \mathbf{N} &= ??\mathbf{E}_y, \\ \left(I_{zz} + mR^2\right)\ddot{\theta} &= ??? + ????. \end{aligned} \tag{7}$$

For full credit, supply the missing terms.

(d) *(5 Points)* Starting from the work-energy theorem, prove that the change in total energy of the rolling rigid body is equal to the work done by the applied moment $M_a\mathbf{E}_z$. For full credit, provide an expression for the total energy E in terms of m, h, x, I_{zz}, R, g, ϕ, and $\dot{\theta}$.

(e) *(6 Points)* Suppose the body is rolling down the incline and that the applied torque is used to brake the wheel (i.e., $M_a > 0$ and $\dot{x} > 0$). For a given inclination angle ϕ and coefficient of friction μ_s what is the maximum torque M_a^{max} that can be applied before the body will start to slide? For full credit, provide an expression for the maximum torque in terms of m, I_{zz}, R, g, ϕ, and μ_s.

Question 2

Tipping Points (20 Points)

In the bottling plant for your favorite beverage, a conveyer belt transports aluminum cans up an incline. A critical component of the design process for the conveyer belt is to determine how fast it can accelerate/decelerate the cans without the cans tipping over. A can is modeled as a rigid body of mass m, moment of inertia relative to its center of mass C of I_{zz} and having two contact points A and B with the conveyer belt. During its motion, a vertical gravitational force acts on the can and the center of mass C of the can has a position vector

$$\bar{\mathbf{x}} = x\mathbf{E}_x + \frac{H}{2}\mathbf{E}_y. \tag{8}$$

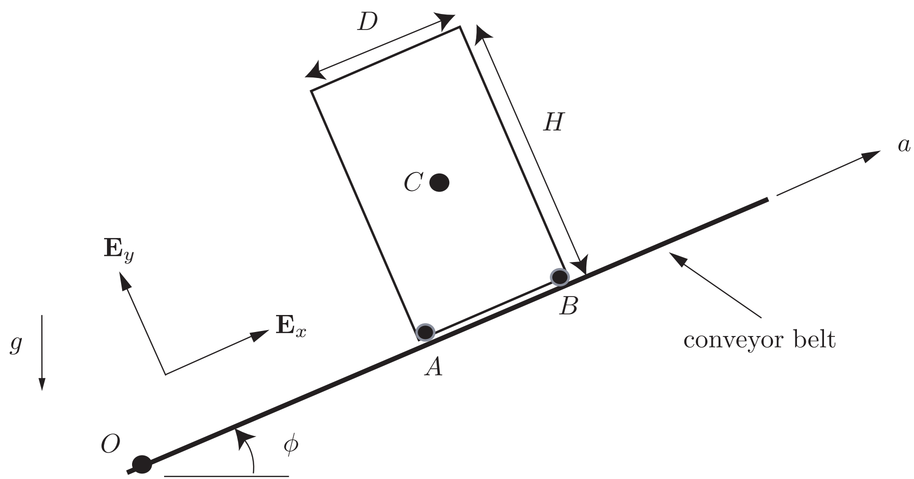

Figure 2: *A rigid body being transported with an acceleration $a\mathbf{E}_x$ by a conveyer belt.*

(a) *(5 Points)* Establish expressions for the angular momentum $\mathbf{H}_O$ and kinetic energy T for the rigid body assuming that it is not rotating.

(b) *(5 Points)* Supposing that friction forces $\mathbf{F}_A = F_A\mathbf{E}_x$ and $\mathbf{F}_B = F_B\mathbf{E}_x$ act at A and B respectively, draw a free–body diagram of the rigid body.

(c) *(5 Points)* Using balances of linear and angular momentum, assuming that both A and B are in contact with the conveyer belt, and that the acceleration of the conveyer belt is a, show that the normal forces acting at A and B are, respectively,

$$\begin{aligned} \mathbf{N}_A &= \left(\tfrac{mg}{2}\cos(\phi) + \tfrac{H}{2D}\left(ma + mg\sin(\phi)\right)\right)\mathbf{E}_y, \\ \mathbf{N}_B &= \left(\tfrac{mg}{2}\cos(\phi) - \tfrac{H}{2D}\left(ma + mg\sin(\phi)\right)\right)\mathbf{E}_y. \end{aligned} \tag{9}$$

(d) *(5 Points)* To prevent cans from tipping over either from by conveyer belt accelerating ($a > 0$) too quickly or decelerating ($a < 0$) too rapidly, show that

$$-\left(\frac{D}{H}\cos(\phi) + \sin(\phi)\right) \le \frac{a}{g} \le \left(\frac{D}{H}\cos(\phi) - \sin(\phi)\right). \tag{10}$$

Question 3

A Particle Colliding with a Rigid Body (30 Points)

As shown in Figure 3, a particle of mass m_1 is traveling with a velocity $-v_0\mathbf{E}_y$ when it collides with a stationary rigid body of mass m_2. The rigid body of m_2 is hinged at its center of mass at O and has a moment of inertia relative to its center of mass of I_{zz}. The rigid body is suspended with the help of a torsional spring which exerts a moment $-K\theta$ on this body. Following the collision, the particle of mass m_1 adheres to the rigid body and the composite body is free to rotate about O.

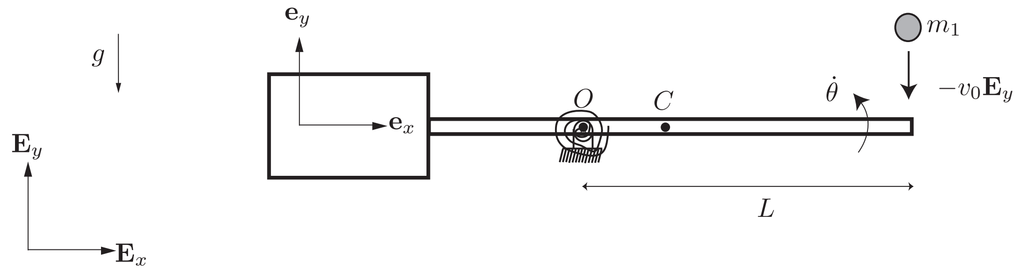

Figure 3: *A particle of mass m_1 collides with a body of mass m_2. The basis vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{E}_z\}$ corotate with the body and the point C is the center of mass of the composite body (of mass $m_1 + m_2$) consisting of the rigid body and the particle.*

(a) *(5 Points)* Assuming that the particle adheres to the end of the rod following the collision, show that the velocity vector $\bar{\mathbf{v}}$ of the center of mass C of the composite body following the collision has the representation

$$\bar{\mathbf{v}} = \left(\frac{m_1}{m_1 + m_2}\right) L\dot{\theta}\mathbf{e}_y. \tag{11}$$

(b) *(5 Points)* Assuming that the particle adheres to the end of the rod following the collision, show that the angular velocity $\boldsymbol{\omega} = \dot{\theta}_0\mathbf{E}_z$ of the composite body immediately following the collision is

$$\boldsymbol{\omega} = -\left(\frac{m_1 L v_0}{I_{zz} + m_1 L^2}\right)\mathbf{E}_z. \tag{12}$$

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(c) *(5+5+5+5 Points)* Consider the motion of the composite body following the collision.

1. Draw a free body diagram of the composite body.
2. Establish the differential equation governing the motion of the body:

$$?\ddot{\theta} = -K\theta + ?? \tag{13}$$

   For full credit supply the missing terms.
3. Establish an expression for the total energy of the composite body and show that the energy E is conserved.
4. Determine the minimum speed v_0 required to ensure that the composite body will achieve a vertical orientation ($\theta = -90°$) following the impact.

Question 4

Balancing a Wheel (20 Points)

As shown in Figure 4, a wheel of mass m_1, radius R, and moment of inertia relative to its center of mass C of I_{zz} is spun about an axle through its center of mass. A mass m_2 is placed on the wheel at a distance h away from the center of the wheel and this mass induces an imbalance. In the sequel, the inertia and mass of the shaft are ignorable.

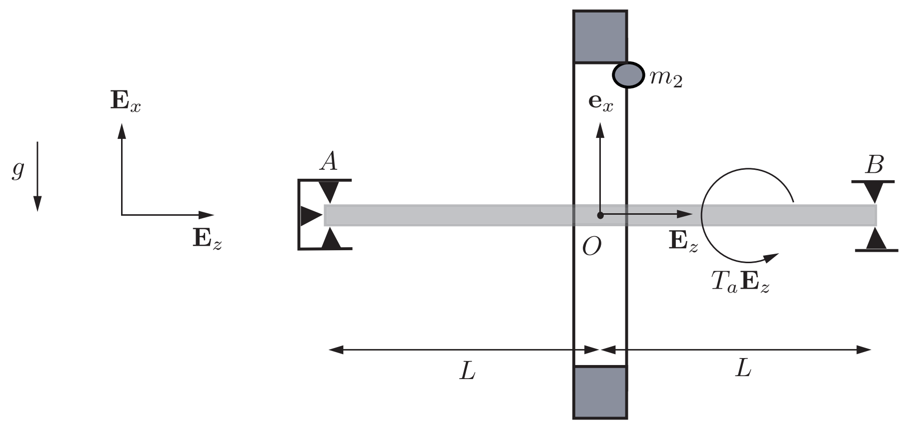

Figure 4: An imbalanced system of mass $m_1 + m_2$ which is free to rotate about the $\mathbf{E}_z$ axis and is supported by bearings at A and B. An applied torque $T_a\mathbf{E}_z$ acts on the assembly.

The center of mass C of the wheel is stationary and coincident with the fixed point O shown in Figure 4. The angular momentum of the wheel relative to its center of mass C is

$$I_{zz}\dot{\theta}\mathbf{E}_z, \tag{14}$$

and the position vector of the particle of mass m_2 relative to O is

$$\mathbf{x}_2 = h\mathbf{e}_x + d\mathbf{E}_z. \tag{15}$$

(a) *(5 Points)* Show that the angular momentum $\mathbf{H}_O$ of the system relative to O has the representation

$$\mathbf{H}_O = \left(I_{zz} + m_2h^2\right)\dot{\theta}\mathbf{E}_z - m_2hd\dot{\theta}\mathbf{e}_x. \tag{16}$$

(b) *(5 Points)* Draw a free-body diagram of the system and compute $\mathbf{M}_O$.

(c) *(2 Points)* Show that the angular speed of the shaft is governed by the equation

$$\left(I_{zz} + m_2h^2\right)\ddot{\theta} = T_a + m_2gh\sin(\theta). \tag{17}$$

(d) *(6 Points)* Assuming that $\dot{\theta} = \omega_0$ is constant, *Verify* that the following expressions for the bearing forces satisfy the balances of linear and angular momenta:

$$\begin{aligned}
\mathbf{R}_A &= \left(\frac{m_1+m_2}{2}\right)g\mathbf{E}_x + \left(\frac{m_2dh\omega_0^2}{2L}\right)\mathbf{e}_x - \left(\frac{m_2d}{2L}\right)g\mathbf{E}_x - \frac{m_2h}{2}\omega_0^2\mathbf{e}_x,\\
\mathbf{R}_B &= \left(\frac{m_1+m_2}{2}\right)g\mathbf{E}_x - \left(\frac{m_2dh\omega_0^2}{2L}\right)\mathbf{e}_x + \left(\frac{m_2d}{2L}\right)g\mathbf{E}_x - \frac{m_2h}{2}\omega_0^2\mathbf{e}_x.
\end{aligned} \tag{18}$$

(e) *(2 Points)* Give the position vector relative to O of the location on the wheel where you would place a bead of mass m_2 to remove the imbalance.

Solution to Question 1

The Braking of a Rolling Rigid Body (30 Points)

Question 2

Tipping Points (25 Points)

Question 3

A Particle Colliding with a Rigid Body (30 Points)

Question 4
Balancing a Wheel (20 POINTS)