

Final Examination
Thursday December 15, 2005
5:00 pm to 8:00 pm
277 Cory Hall

Closed Books and Closed Notes
Each Question is Worth 25 Points

Useful Formulae

Recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment \mathbf{M}_p is

$$\dot{T} = \sum_{i=1}^K \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \quad (1)$$

Here, \mathbf{v}_i is the velocity vector of the point where the force \mathbf{F}_i is applied.

Question 1
A Driving Exercise

As shown in Figure 1, an automobile is being driven down an incline. Here, we model the entire vehicle as a uniform rigid body of mass m , length $2L$ and height $2h$ whose angular momentum vector \mathbf{H} and angular velocity vector $\boldsymbol{\omega}$ are

$$\mathbf{H} = I_{zz}\boldsymbol{\omega}\mathbf{E}_z, \quad \boldsymbol{\omega} = \omega\mathbf{E}_z = \dot{\theta}\mathbf{E}_z. \quad (2)$$

We also assume that the position vector of the center of mass C of the car is

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y\mathbf{E}_y. \quad (3)$$

The dimensions of the wheels are assumed to be negligible.

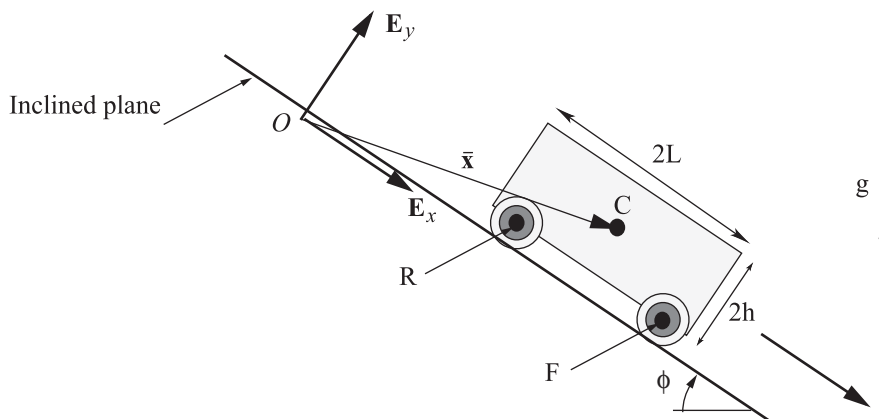


Figure 1: Schematic of an automobile of mass m moving down an inclined plane. The point F on the automobile has a position vector $\mathbf{x}_F = x_F\mathbf{E}_x + y_F\mathbf{E}_y$.

(a) (5 Points) Show that the kinetic energy, and angular momentum (relative to a point F on the vehicle) are

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_{zz}\dot{\theta}^2, \quad \mathbf{H}_F = \left(I_{zz}\dot{\theta} + m(x - x_F)\dot{y} - m(y - y_F)\dot{x} \right) \mathbf{E}_z. \quad (4)$$

(b) (5 Points) Assuming that a traction force \mathbf{F}_t only acts on the front wheels and that both sets of wheels are in contact with the ground, draw a free-body diagram of the car.

(c) (7 Points) Using balances of linear and angular momentum and assuming that both sets of wheels are in contact with the ground, show that the equations governing the motion of the car is

$$m\ddot{x} = \mathbf{F}_t \cdot \mathbf{E}_x + mg \sin(\phi), \quad (5)$$

and that the two normal forces acting on the sets of front and rear wheels are

$$\begin{aligned} \mathbf{N}_F &= \frac{1}{2} \left(mg \cos(\phi) - \frac{h}{L}(m\ddot{x} - mg \sin(\phi)) \right) \mathbf{E}_y, \\ \mathbf{N}_R &= \frac{1}{2} \left(mg \cos(\phi) + \frac{h}{L}(m\ddot{x} - mg \sin(\phi)) \right) \mathbf{E}_y. \end{aligned} \quad (6)$$

(d) (8 Points) Assuming that both sets of wheels are in contact with the ground, verify that $\dot{\mathbf{H}}_F \neq \mathbf{0}$ and that $\dot{\mathbf{H}}_F = \mathbf{M}_F - \mathbf{v}_F \times \mathbf{G}$ where \mathbf{M}_F is the resultant moment relative to F .

Question 2
A Rolling Rigid Body

As shown in Figure 2, a body of mass m_1 is rigidly connected to a circular hoop of mass m_2 and radius R . The composite rigid body has a mass $m = m_1 + m_2$ and its center of mass \bar{X} is located at a distance h from the center C of the disk. The angular momentum of the body about its center of mass \bar{X} is

$$\mathbf{H} = I_{zz}\boldsymbol{\omega} \quad (7)$$

where $\boldsymbol{\omega} = \omega\mathbf{E}_z$ is the angular velocity of the rigid body.

Figure 2: A rigid body of mass m rolling on a rough horizontal plane.

(a) (5 Points) Starting from the result $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ for any two points on a rigid body, show that the velocity vectors of the center of mass \bar{X} and the center C of the hoop have the representations

$$\bar{\mathbf{v}} = -R\dot{\theta}\mathbf{E}_x + h\dot{\theta}\mathbf{e}_y, \quad \mathbf{v}_C = \dot{x}\mathbf{E}_x, \quad (8)$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the rigid body.

(b) (5 Points) Draw a free-body diagram of the rolling rigid body.

(c) (8 Points) Using balances of linear and angular momentum, show that

$$\begin{aligned} \mathbf{F}_f &= -\left(mR\ddot{\theta} + mh\dot{\theta}^2 \cos(\theta) + mh\ddot{\theta} \sin(\theta)\right)\mathbf{E}_x, \\ \mathbf{N} &= \left(mg - mh\dot{\theta}^2 \sin(\theta) + mh\ddot{\theta} \cos(\theta)\right)\mathbf{E}_y, \\ I_{tot}\ddot{\theta} &= -mgh \cos(\theta) - mRh\dot{\theta}^2 \cos(\theta), \end{aligned} \quad (9)$$

where

$$I_{tot} = I_{zz} + mh^2 + mR^2 + 2mhR \sin(\theta). \quad (10)$$

(d) (3 Points) Write down expressions for the kinetic energy T and total energy E of the rolling rigid body.

(e) (4 Points) Starting from the work-energy theorem, prove that E is conserved.

Question 3

A Collision

As shown in Figure 3, a particle of mass m_1 has a velocity vector $\mathbf{v}_1 = -v_0\mathbf{E}_x$ at the instant before it collides with a rigid plate. The point at which the collision occurs has a position vector $\bar{\mathbf{x}}_2 + a\mathbf{e}_x$, where $\bar{\mathbf{x}}_2$ is the position vector of the center of mass C_2 of the plate.

Figure 3: A plate of mass m_2 , length $2a$ and height $2b$ is free to move on a horizontal plane. A particle of mass m_1 collides with the rigid plate at the instant shown, and coalesces with it after the impact.

The plate is free to move on a horizontal plane and its angular momentum relative to its center of mass is $\mathbf{H}_2 = m_2k^2\dot{\theta}\mathbf{E}_z$, where $\dot{\theta}\mathbf{E}_z$ is its angular velocity vector. Just prior to the collision, the center of mass of the plate is stationary.

(a) (5 Points) Show that the position vector $\bar{\mathbf{x}}$ of the center of mass C of the system at any instant after the collision has the representations

$$\bar{\mathbf{x}} = \mathbf{x}_1 - \left(\frac{m_2}{m_1 + m_2}\right)a\mathbf{e}_x = \bar{\mathbf{x}}_2 + \left(\frac{m_1}{m_1 + m_2}\right)a\mathbf{e}_x. \quad (11)$$

(b) (5 Points) Show that the velocity vector of the center of mass of the system immediately after the collision is

$$(\bar{\mathbf{v}})_{after} = -\frac{m_1}{m_1 + m_2}v_0\mathbf{E}_x. \quad (12)$$

(c) (8 Points) Show that the angular momentum of the system (relative to the center of mass of the system) immediately before, and after the collision are

$$(\mathbf{H})_{before} = \left(m_2k^2\omega_0 + \left(\frac{m_1m_2}{m_1 + m_2}\right)av_0\sin(\theta_0)\right)\mathbf{E}_z, \quad (\mathbf{H})_{after} = I_{zz}\dot{\theta}\mathbf{E}_z, \quad (13)$$

where, θ_0 is the initial value of θ , ω_0 is the initial angular speed of the plate and

$$I_{zz} = m_2k^2 + \left(\frac{m_1m_2}{m_1 + m_2}\right)a^2. \quad (14)$$

(d) (7 Points) Assuming that $\theta_0 = 0$, show that the following energy is lost as a result of the collision:

$$(T)_{before} - (T)_{after} = \frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)\left(v_0^2 + \left(\frac{m_2k^2}{I_{zz}}\right)a^2\omega_0^2\right). \quad (15)$$

Question 4

A Rigid Body Rotating about a Fixed Point

As shown in Figure 4, a slender rod of mass m_1 , moment of inertia I_{zz} and length $2L$ is pivoted at a fixed point O using a pin-joint. The rod is free to rotate about \mathbf{E}_z . In addition to a vertical gravitational force $-m_1g\mathbf{E}_z$, a spring force \mathbf{F}_s acts on the rod. The force \mathbf{F}_s is due to a linear spring of stiffness K and unstretched length L_0 . One end of this spring is attached to the end of the rod, while the other end is attached to a particle of mass m_2 .

Figure 4: A rigid rod of length $2L$ is free to rotate about a fixed point O . At the end of the rod, a particle of mass m_2 hangs freely by a linear spring.

(a) (5 Points) If the position vector of the center of mass C of the rod is $L\mathbf{e}_x$, show that the linear momentum, angular momentum and kinetic energy of the rigid body are

$$\mathbf{G} = m_1L\dot{\theta}\mathbf{e}_y, \quad \mathbf{H}_O = (I_{zz} + m_1L^2)\dot{\theta}\mathbf{E}_z, \quad T = \frac{1}{2}(I_{zz} + m_1L^2)\dot{\theta}^2. \quad (16)$$

(b) (5 Points) Draw free-body diagrams of the rigid body and the particle. Give a clear expression for the spring force.

(c) (5 Points) Show that the equation governing the motion of the rigid body and the position vector \mathbf{x}_2 of the particle are

$$\begin{aligned} (I_{zz} + m_1L^2)\ddot{\theta} &= (2L\mathbf{e}_x \times \mathbf{F}_s) \cdot \mathbf{E}_z, \\ m_2\ddot{\mathbf{x}}_2 &= -\mathbf{F}_s - m_2g\mathbf{E}_z. \end{aligned} \quad (17)$$

(d) (5 Points) Using balances of linear and angular momentum, determine expressions for the reaction force at O and the moment ensuring that $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$.

(e) (5 Points) Prove the total energy of the system consisting of the particle and the rigid body is conserved. In your solution, give a clear expression for the total energy E of the system.