Math 55 midterm exam, April 4, 2019

Problem	1	2	3	4	5	6	Total
Points	6	5	6	5	6	6	34

You all acted with honesty, integrity, and respect for others.

1a. For $n \geq 0$, determine the number of bit strings of length n that do not contain the string `01`.

b. Let b_n be the number of bit strings of length n that do not contain `000`. Show that $b_0 = 1$, $b_1 = 2$, $b_2 = 4$ and

$$b_{n+3} = b_{n+2} + b_{n+1} + b_n$$

for $n \geq 0$.

2. Math 55 students Alice and Bob announce their RSA public keys as $(n, 13)$ and $(n, 40)$; because they are good friends, they use the same modulus n. After learning that Charlie employed RSA to send the same message to Alice and Bob, Eve succeeds at retrieving the encrypted texts that Charlie sent to the two recipients. How can Eve recover Charlie's plain text from the two encrypted texts?

3a. How many ways are there to distribute six red hats, six blue hats and six gold hats to a group of 18 students if each student receives one hat? [The hats are identical except for their colors.]

b. How many ways are there to carry out the task in part (a) if the two identical twins in the class are to receive the same color hat?

4. Let $\alpha = (1 + \sqrt{5})/2$. Prove that the nth Fibonacci number is less than α^{n-1} for all $n \geq 2$.

5. Alice buys a bag of 12 coins. Four of these are biased coins that come up heads 3/4 of the time; the other eight are fair coins. Sylvia reaches into the bag, pulls out a coin at random and tosses it. Sylvia's coin comes up heads! What is the probability that she pulled out a top-heavy coin?

6a. Given that a poker hand contains at least three of the four aces in a deck of cards, what is the probability that it contains all four of the aces?

b. An urn contains 20 red balls, 20 green balls and 20 blue balls. Three of the 60 balls are removed at random. What is the expected number of green balls that have been removed from the urn?