Stat153 Midterm Exam 2. (November 9, 2010)

Name:..

Student ID:..

This is an open-book exam: you can use any material you like. Exam papers will be handed out at 12:40, the exam will go from 12:45 to 1:55. Answer all three questions. Each part of each question has a percentage written next to it: the percentage of the grade that it constitutes.

1. Let $\{X_t\}$ be a stationary time series with spectral density f_x. Suppose that the time series $\{Y_t\}$ is obtained by mixing a proportion $\alpha \in [0,1]$ of this time series with a proportion $1-\alpha$ of the time series delayed by k time steps:

$$Y_t = \alpha X_t + (1-\alpha) X_{t-k}.$$

   (a) Show that the spectral density of $\{Y_t\}$ is

$$f_y(\nu) = \big(\alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)\cos(2\pi\nu k)\big)\, f_x(\nu).$$

(10%)

$$\begin{aligned}
f_y(\nu) &= \sum_{h=-\infty}^{\infty} \gamma_y(h) e^{-2\pi\nu h} \\
&= \sum_{h=-\infty}^{\infty} \big(\alpha^2\gamma_x(h) + (1-\alpha)^2\gamma_x(h) + \alpha(1-\alpha)\mathsf{Cov}(X_t, X_{t+h-k}) + \alpha(1-\alpha)\mathsf{Cov}(X_{t-k}, X_{t+h})\big)\, e^{-2\pi\nu h} \\
&= \big(\alpha^2 + (1-\alpha)^2 + \alpha(1-\alpha)(e^{-2\pi\nu k} + e^{2\pi\nu k})\big)\, f_x(\nu) \\
&= \big(\alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)\cos(2\pi\nu k)\big)\, f_x(\nu).
\end{aligned}$$

(b) If $\{X_t\}$ is white, $k = 3$ and $\alpha = 1/2$, show that the spectral density of $\{Y_t\}$ is periodic and calculate its period. **(10%)**

We have $f_x(\nu) = \sigma^2$, so

$$f_y(\nu) = \sigma^2 \left(\alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)\cos(2\pi\nu k)\right) = \sigma^2 \left(1/2 + 1/2\cos(6\pi\nu)\right).$$

Since $\cos(2\pi(\nu + 1/k)k) = \cos(2\pi\nu k)$, this is periodic with period $1/k = 1/3$.

2. Consider the stationary time series $\{X_t\}$ defined by

$$X_t = 1/(1.01)^3 X_{t-3} + W_t + 0.4W_{t-1},$$

where $\{W_t\} \sim WN(0, \sigma_w^2)$.

(a) Express X_t in the form

$$X_t = \psi(B)W_t,$$

where $\psi(B)$ is a rational function (ratio of polynomials) of the back-shift operator B. Specify the rational function ψ, and show that it has poles at 1.01, $1.01e^{i2\pi/3}$, and $1.01e^{-i2\pi/3}$ and a zero at -2.5. **(10%)**

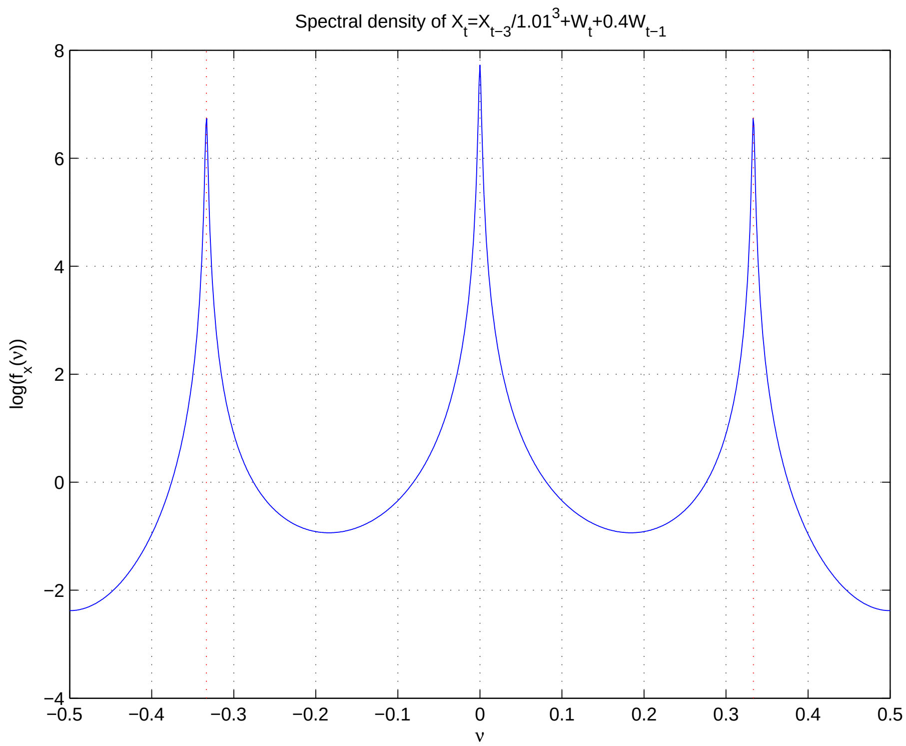
We can write $(1 - B^3/(1.01)^3)X_t = (1 + 0.4B)W_t$, and so $\psi(B) = (1 + 0.4B)/(1 - B^3/(1.01)^3)$. The poles of $\psi(B)$ are the roots of the polynomial $1 - z^3/(1.01)^3$, which satisfy

$$|z| = 1.01, 3\arg(z) \in \{0, \pm 2\pi, \ldots\}.$$

The zero of $\psi(B)$ is the root of the polynomial $1 + 0.4z$, which is $z = -1/0.4 = -2.5$.

(b) Using your answer to part (a), make a rough sketch of the spectral density of $\{X_t\}$. Explain the origin of the various features of the spectral density. **(10%)**

The sketch should have a large peak near $\nu = 1/3$, since at that point $e^{2\pi i\nu}$ is closest to the pole at $1.01e^{2\pi i/3}$. Because of the pole at 1.01 and the zero, it should also have a peak at the origin $\nu = 0$, and a minimum at $\nu = 1/2$. Here's what the true spectral density looks like:



(c) It should be clear from your sketch that the realizations of $\{X_t\}$ will exhibit approximately oscillatory behavior. What is the period of these oscillations? **(10%)**

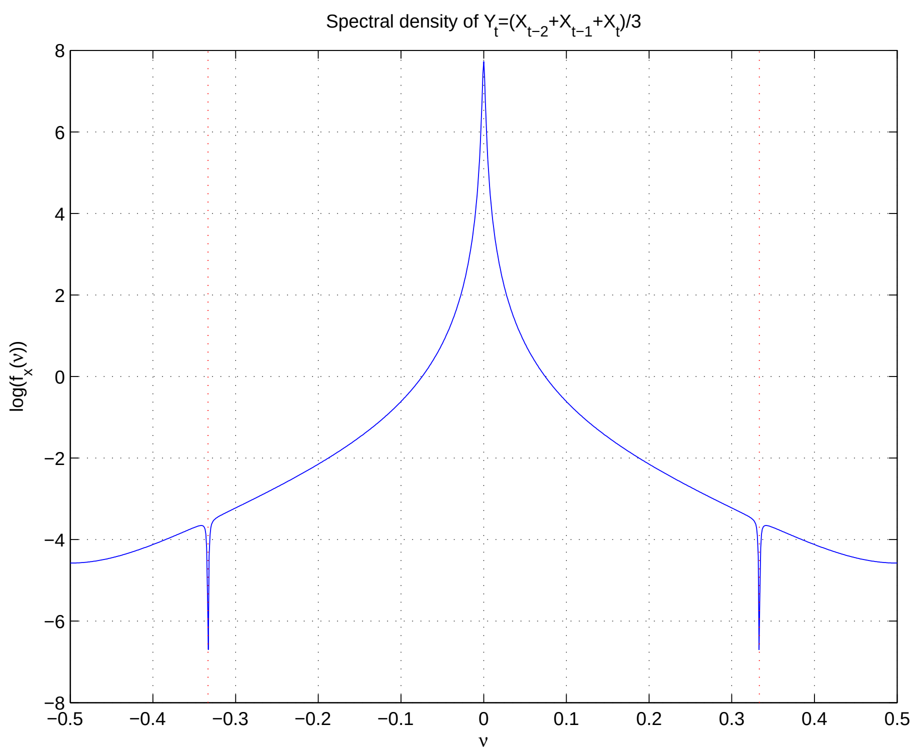
The frequency of the oscillations would be at the peak of the spectral density, $\nu = 1/3$. Thus, the period is 3 samples.

Suppose that we pass the time series $\{X_t\}$ through a linear filter, to obtain the series $\{Y_t\}$,

$$Y_t = \frac{1}{3}\left(X_{t-2} + X_{t-1} + X_t\right).$$

(d) By writing Y_t in the form $Y_t = \xi(B)W_t$ for some rational function $\xi(B)$, make a rough sketch of the spectral density of $\{Y_t\}$. Explain the origin of the various features of the spectral density. Comment on the effect of the filter on the oscillatory behavior. **(15%)**

We have $Y_t = 1/3(1+B+B^2)X_t = 1/3(1+B+B^2)(1+0.4B)/(1-B^3/(1.01)^3)W_t$. Thus, the spectral density of $\{X_t\}$ at a frequency ν is multiplied by the factor $|1+z+z^2|^2/9$, where $z = e^{2\pi i\nu}$. Notice that z^2+z+1 has roots at $-1/2 \pm i\sqrt{3}/2 = e^{\pm 2\pi i/3}$. These zeros will remove the oscillatory component, since they are at the frequency $\nu = 1/3$ of the peak of $f_x(\nu)$. The spectral density looks like this:



3. Suppose that a certain time series $\{Y_t\}$ has a quadratic trend component, a seasonal component, and a stationary component:

$$Y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + g(t) + X_t,$$

where $\alpha_0, \alpha_1, \alpha_2$ are non-zero constants, $g(t)$ is a non-constant periodic function of t, with period 12 (that is, for all t, $g(t+12) = g(t)$), and $\{X_t\}$ is a stationary time series with spectral density $f_x(\nu)$.

(a) Show that $\{Y_t\}$ is not stationary. **(10%)**

Since the function $\alpha_1 t + \alpha_2 t^2$ is not constant and not periodic, and $g(t)$ is periodic, we must have that the function $\alpha_1 t + \alpha_2 t^2 + g(t)$ is non-constant. Thus, $\mathbb{E}Y_t$ depends on t, which implies $\{Y_t\}$ is not stationary.

(b) Suggest linear transformations that could be applied to $\{Y_t\}$ that would result in a stationary time series. **(10%)**

To remove the periodic component, we could apply a seasonal difference with period 12; to remove the quadratic component, we could take second differences. Thus, we obtain the time series

$$Z_t = (1 - B)^2(1 - B^{12})Y_t.$$

(c) Show that when you apply the linear transformations of part (3b), the resulting time series (call it $\{Z_t\}$) is stationary.
Express $f_z(\nu)$, the spectral density of $\{Z_t\}$, in terms of α_0, α_1, α_2, $g(\cdot)$, and $f_x(\nu)$. **(15%)**

To see that $\{Z_t\}$ is stationary, first notice that $(1 - B^{12})$ removes the periodic component:

$$\begin{aligned}(1 - B^{12})Y_t &= \alpha_1(t - (t - 12)) + \alpha_2(t^2 - (t - 12)^2) + (g(t) - g(t - 12)) + (1 - B^{12})X_t \\ &= 12\alpha_1 + \alpha_2(24t - 144) + (1 - B^{12})X_t.\end{aligned}$$

Second, taking second differences removes the trend component:

$$\begin{aligned}Z_t = (1 - B)^2(1 - B^{12})Y_t &= (1 - B)\left(24\alpha_2(t - (t - 1)) + (1 - B)(1 - B^{12})X_t\right) \\ &= (1 - B)^2(1 - B^{12})X_t.\end{aligned}$$

Clearly, $\{Z_t\}$ is obtained from the stationary time series $\{X_t\}$ by taking linear combinations of time-shifted versions. Since these operations preserve stationarity, $\{Z_t\}$ is stationary.

From the expression above, the spectral density of $\{Z_t\}$ can be written as

$$f_z(\nu) = |1 - e^{2\pi i\nu}|^4 |1 - e^{24\pi i\nu}|^2 f_x(\nu).$$