Stat153 Midterm Exam 2. (November 9, 2010)

Name:...

Student ID:..

This is an open-book exam: you can use any material you like. Exam papers will be handed out at 12:40, the exam will go from 12:45 to 1:55. Answer all three questions. Each part of each question has a percentage written next to it: the percentage of the grade that it constitutes.

1. Let $\{X_t\}$ be a stationary time series with spectral density f_x. Suppose that the time series $\{Y_t\}$ is obtained by mixing a proportion $\alpha \in [0, 1]$ of this time series with a proportion $1 - \alpha$ of the time series delayed by k time steps:

$$Y_t = \alpha X_t + (1 - \alpha) X_{t-k}.$$

   (a) Show that the spectral density of $\{Y_t\}$ is

$$f_y(\nu) = \big(\alpha^2 + (1 - \alpha)^2 + 2\alpha(1 - \alpha)\cos(2\pi\nu k)\big)\, f_x(\nu).$$

(10%)

(b) If $\{X_t\}$ is white, $k = 3$ and $\alpha = 1/2$, show that the spectral density of $\{Y_t\}$ is periodic and calculate its period. **(10%)**

2. Consider the stationary time series $\{X_t\}$ defined by

$$X_t = 1/(1.01)^3 X_{t-3} + W_t + 0.4W_{t-1},$$

where $\{W_t\} \sim WN(0, \sigma_w^2)$.

(a) Express X_t in the form

$$X_t = \psi(B)W_t,$$

where $\psi(B)$ is a rational function (ratio of polynomials) of the back-shift operator B. Specify the rational function ψ, and show that it has poles at 1.01, $1.01e^{i2\pi/3}$, and $1.01e^{-i2\pi/3}$ and a zero at -2.5. **(10%)**

(b) Using your answer to part (a), make a rough sketch of the spectral density of $\{X_t\}$. Explain the origin of the various features of the spectral density. **(10%)**

(c) It should be clear from your sketch that the realizations of $\{X_t\}$ will exhibit approximately oscillatory behavior. What is the period of these oscillations? **(10%)**

Suppose that we pass the time series $\{X_t\}$ through a linear filter, to obtain the series $\{Y_t\}$,

$$Y_t = \frac{1}{3}\left(X_{t-2} + X_{t-1} + X_t\right).$$

(d) By writing Y_t in the form $Y_t = \xi(B)W_t$ for some rational function $\xi(B)$, make a rough sketch of the spectral density of $\{Y_t\}$. Explain the origin of the various features of the spectral density. Comment on the effect of the filter on the oscillatory behavior. **(15%)**

3. Suppose that a certain time series $\{Y_t\}$ has a quadratic trend component, a seasonal component, and a stationary component:

$$Y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + g(t) + X_t,$$

where $\alpha_0, \alpha_1, \alpha_2$ are non-zero constants, $g(t)$ is a non-constant periodic function of t, with period 12 (that is, for all t, $g(t+12) = g(t)$), and $\{X_t\}$ is a stationary time series with spectral density $f_x(\nu)$.

   (a) Show that $\{Y_t\}$ is not stationary. **(10%)**

(b) Suggest linear transformations that could be applied to $\{Y_t\}$ that would result in a stationary time series. **(10%)**

(c) Show that when you apply the linear transformations of part (3b), the resulting time series (call it $\{Z_t\}$) is stationary.
Express $f_z(\nu)$, the spectral density of $\{Z_t\}$, in terms of α_0, α_1, α_2, $g(\cdot)$, and $f_x(\nu)$. **(15%)**