1. (16pts) Consider a sequence of n independent coin tosses where $\Pr[\texttt{Heads}] = 1/4$ and $\Pr[\texttt{Tails}] = 3/4$. Let X be the number of heads.

   (a) What is the probability of getting $X = k$.

   (b) Apply Markov's bound to $\Pr[X \geq n/2]$.

   (c) Apply Chebyshev's bound to $\Pr[X \geq n/2]$.

   (d) Compute the moment generating function for X.

2. (15pts) Suppose you throw $n/2$ balls into n bins, where n is even. What is the probability that exactly $n/2$ bins are empty?

   (a) Compute this exactly.

   (b) Use the Poisson approximation to give an upper bound on this value.

   (c) Show that your bounds in part (a) is exponentially tighter than the bound in part (b) for large n.

3. (15pts) Consider the number of 3-cliques in a $G_{n,p}$ random graph, denoted by X.

(a) What is the expected number of 3-cliques in a $G_{n,p}$ random graph?

(b) Show that if $p = f(n) = o(1/n)$ then for any $\varepsilon > 0$ there exists some n sufficiently large such that the probability a clique will exist is less than ε.

(c) Show that $p = f(n) = \omega(1/n)$ then for any $\varepsilon > 0$ there exists some n sufficiently large such that the probability a clique will not exist is less than ε.

4. (12pts) Consider a k-SAT formula with n literals and m clauses, where each literal is contained in at most 3 consecutive clauses, i.e. the clauses are ordered 1 to m and a specific literal can only be in 3 in a row, such as clauses $i, i+1, i+2$. In this problem you will apply the Lovasz local lemma.

   (a) Define the probability space and the set of bad events.

   (b) Describe the dependency graph (construct the vertex set and edge set).

   (c) For what values of k must there exist a solution to the formula according to the Lovasz local lemma?

5. (12pts) Consider a fair lazy random walk on $\{0, \cdots, n\}$ with reflecting boundaries, i.e. at every state i except 0 and n, the next step is chosen from $\{i, i+1, i-1\}$ with equal probability; while at state 0 it is 1 or 0 (unchanged) and at state n it is $n-1$ or n, both with equal probabilities.

(a) For $n = 3$, draw the graph of this Markov Chain, including the transition probabilities.

(b) For $n = 3$, explicitly write out the transition matrix of this Markov Chain.

(c) For arbitrary n compute the stationary distribution of this Markov Chain.

(d) If the Markov chain starts at state 0, show that the probability of not returning to state 0 within $\frac{3}{2}n^2$ steps is less than $1/n$.

6. (12pts) Consider a d-regular graph G with n vertices (d-regular means that every vertex has degree d). A dominating set S is a subset of the vertices such that every vertex in G is either in S or a neighbor of a vertex in S.

   (a) Show that $|S| \geq \frac{n}{1+d}$.

   (b) Suppose $d \geq 4 \ln n$, and we choose a set S of vertices at random, where each vertex is chosen with probability $p = \frac{2 \ln n}{d+1}$. Show that the probability that S is a dominating set is at least $1 - 1/n$. (Hint: if $x > 0$, $(1 - 1/x)^x < 1/e$.)

   (c) Construct an algorithm which improves on the approach in part (b) and produces a dominating set with $E[|S|] \leq n\frac{1+\ln(d+1)}{d+1}$. (Hint: sample and modify.)

7. (12pts) Suppose that S and T are stopping times for the sequence $\{Z_n : n \geq 0\}$. Which of the following are necessarily stopping times for the sequence $\{Z_n : n \geq 0\}$? Justify your answers.

   (a) $S + T$.

   (b) $\max(S, T) - \min(S, T)$.

   (c) S^2.