Name	
1a	
1b	
2a	
2b	
2c	
3a	
3b	
4a	
4b	
Total	

(1a) Suppose x_n is a sequence of real numbers defined by $x_0 = 1$ and

$$x_{n+1} = 2x_n - \frac{1}{2}x_n^2 = g(x_n).$$

Assume $x_n \to x$ for some x as $n \to \infty$. Show that

$$|x_{n+1} - x| \leq \frac{1}{2}|x_n - x|^2.$$

(1b) In floating point arithmetic, x_n is approximated by y_n satisfying

$$y_{n+1} = \mathrm{fl}(x_{n+1}) = (2y_n - \frac{1}{2}y_n^2(1+\delta_n))(1+\delta_n')$$

where division and multiplication by 2 are exact, $|\delta_n| \leq \epsilon$, and $|\delta_n'| \leq \epsilon$. Show that

$$|y_{n+1} - x| \leq \frac{1}{2}|y_n - x|^2 + 4\epsilon + O(\epsilon^2)$$

and describe the behavior of $|y_n - x|$ as $n \to \infty$.

(2a) Define the centered difference quotient by

$$D_h f(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

Use Taylor expansion to show that

$$|D_h f(x) - f'(x)| \leq \frac{M_3}{6} h^2$$

whenever $|f'''(x)| \leq M_3$ for all x.

(2b) Suppose f can be evaluated with relative error bounded by ϵ. Show that floating-point arithmetic with machine epsilon ϵ gives

$$|\mathrm{fl}(D_h f(x)) - D_h f(x)| \leq \frac{6M_0\epsilon}{h} + O(\epsilon^2)$$

whenever $|f(x)| \leq M_0$ for all x.

(2c) Combining (2a) and (2b) and dropping $O(\epsilon^2)$ terms gives an error bound

$$|f'(x) - \mathrm{fl}(D_h f(x))| \leq \frac{M_3}{6} h^2 + \frac{6 M_0 \epsilon}{h} = F(h).$$

Find h (as a function of ϵ, M_0, and M_3) which minimizes $F(h)$ and evaluate the minimum value of $F(h)$.

(3a) Let $p(x)$ be the quadratic polynomial interpolating function values $f(x_1)$, $f(x_2)$ and $f(x_3)$ at equally spaced points $x_1 = -h$, $x_2 = 0$ and $x_3 = h$.

Give a formula for the error $f(x) - p(x)$ which includes a kth derivative $f^{(k)}(\xi)$ evaluated at an unknown point ξ. Explain why the value of k is inevitable.

(3b) For the specific function $f(x) = |x|$ show that the error $f(x) - p(x)$ is $O(h)$ when $|x| \leq h$ and explain the apparent contradiction with (3a).

(4a) Find constants a and b such that the numerical integration rule

$$\int_0^3 f(x)dx = af(0) + bf(2)$$

is exact whenever f is a polynomial of degree 2.

(4b) Assume we know the a and b from question (4a). Write down weights w_1 and w_2 such that

$$\int_0^{3h} g(x)dx = w_1 g(0) + w_2 g(2h) + O(h^4).$$