1A	
1B	
2A	
2B	
3A	
3B	
Total	

1A Let x_n be a sequence of real numbers defined by $x_0 = 5$ and

$$x_{n+1} = 2 + \frac{1}{x_n} = g(x_n).$$

Assume $x_n \to x$ for some $x \geq 2$ as $n \to \infty$. Show that

$$|x_{n+1} - x| \leq \frac{1}{4}|x_n - x|$$

for all n.

1B In floating point arithmetic, x_n is approximated by y_n satisfying

$$y_{n+1} = \mathrm{fl}(x_{n+1}) = \left(2 + \frac{1}{y_n}(1+\delta_n)\right)(1+\delta'_n)$$

where $|\delta_n| \leq \epsilon$ and $|\delta'_n| \leq \epsilon$.
(a) Show that

$$|y_{n+1} - x| \leq \frac{1}{4}|y_n - x| + 3\epsilon + O(\epsilon^2)$$

for all n, and
(b) describe the behavior of y_n as $n \to \infty$.

2A Let $H(x)$ be the cubic polynomial interpolating $f(0)$, $f'(0)$, $f(1)$, and $f'(1)$.
(a) Give a formula for the error $f(x) - H(x)$ which includes the pth-order derivative $f^{(p)}(\xi)$, evaluated at an unknown point ξ.
(b) Specify the value of p and explain why it is inevitable.

2B For the specific function $f(x) = x^4$,
(a) build the divided difference table,
(b) find the Newton form of $H(x)$,
(c) evaluate $H(1/2)$, and
(d) show that your error formula from (2A) is satisfied at $x = 1/2$.

3A

(a) Find constants a, b and c such that the numerical integration rule

$$\int_0^1 f(t)\ dt = af(-1) + bf(0) + cf(1)$$

is exact whenever f is a quadratic polynomial. (Hint: Integrate Lagrange basis polynomials or solve a linear system.)

(b) Find constants a', b' and c' such that the numerical integration rule

$$\int_0^1 f(t)\ dt = a'f(0) + b'f(1) + c'f(2)$$

is exact whenever f is a quadratic polynomial. (Hint: Change variables, integrate Lagrange basis polynomials, or solve a linear system.)

3B Find weights w_0, w_1, w_2 which make the numerical integration rule

$$\int_0^{Nh} f(x)dx = \sum_{j=0}^{N-1} h \int_0^1 f(jh+th)dt = h \sum_{j=0}^{N} w_j f(jh), \qquad w_3 = \cdots = w_{N-3} = 1,$$

accurate to order $O(h^3)$.