Math 55 midterm exam, February 21, 2019

Problem	1	2	3	4	5	6	Total
Points	6	6	5	6	5	6	34

1a. *If S is a subset of a countable set, is S necessarily countable? Explain your answer carefully, outlining a proof or giving a counterexample.*

Yes, S is countable. Suppose that $S \subseteq A$, where A is countable. Then A is either finite or countably infinite. If A is finite, say of size n, then S has no more than n elements and is therefore finite as well. Let us assume now that A is countably infinite. Then, by definition, we can list off the elements of A:

$$A = \{ a_1, a_2, a_3, \ldots \},$$

where the a_i are distinct. We can imagine putting small circles around each element a_i that belongs to S. Then the elements of S are listed: the first element of S is the first element of A with a circle around it, the second element of S is the second circled element, etc. If list stops, there are only a finite number of elements of S. (There might in fact be no circled elements of A; then S is the empty set and has 0 elements!) If the list continues indefinitely, we have counted off all elements of S and see that S is countably infinite.

b. *Suppose that $f : T \to \{ 1, 2, 3, \ldots \}$ is an onto function. Is the set T necessarily countable?*

No, T is not necessarily countable. To see this, it's best to exhibit a specific example where T is uncountable. We can take T to be the set of real numbers and define

$$f(x) = 1 + \lfloor\, |x| \,\rfloor, \qquad x \in T.$$

(In words, the right-hand side is 1 plus the floor of the absolute value of x.) Note that $|x|$ is non-negative, so $\lfloor\, |x| \,\rfloor$ is non-negative as well; the addition of 1 ensures that $f(x)$ is an integer ≥ 1. If n is a natural number, $f(n) = n + 1$; therefore $f(0) = 1$, $f(1) = 2$, etc. Thus f is surjective ("onto"). We have seen in our discussions that T is uncountable.

2. *Using mathematical and logical operators, predicates, and quantifiers (where the domain consists of all integers) express: "The difference of two positive integers is not necessarily positive."*

This is exercise 19b of §1.5 in the book. The solution given by Rosen is:

$$\neg\forall x\forall y\big((x > 0 \wedge y > 0) \rightarrow (x - y > 0)\big).$$

That seems good to me, though I might have put an extra pair of parentheses around $\forall x\forall y\big((x > 0 \wedge y > 0) \rightarrow (x - y > 0)\big)$.

3. *Prove or disprove: if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.*

The statement is incorrect for finite sets and therefore false in general: Suppose that A and B are finite, with n and m elements, respectively. (Weirdly, mathematicians often like to put n before m.) Then $\mathcal{P}(A \times B)$ has 2^{nm} elements but $\mathcal{P}(A) \times \mathcal{P}(B)$ has 2^{n+m} elements. More specifically, if A and B each have one element, then $A \times B$ has one element and $\mathcal{P}(A \times B)$ has two elements; on the other hand, $\mathcal{P}(A)$ and $\mathcal{P}(B)$ each have two elements, so $\mathcal{P}(A) \times \mathcal{P}(B)$ has four elements.

4. *Use the Euclidean algorithm to find the gcd of 39 and 57 and to write the gcd as a linear combination of 39 and 57.*

We divide 39 into 57 and get

$$57 = 1 \cdot 39 + 18, \qquad 18 = 57 - 39.$$

We next divide 18 into 39 and get

$$39 = 2 \cdot 18 + 3, \qquad 3 = 39 - 2 \cdot 18 = 3 \cdot 39 - 2 \cdot 57.$$

We finally divide 3 into 18 and discover that the division is exact (remainder $= 0$). Therefore 3 is the gcd; we have written the gcd as a linear combination above.

5. *Find the smallest non-negative integer satisfying the three congruences*

$$x \equiv \begin{cases} -3 & \textit{mod } 19 \\ -3 & \textit{mod } 20 \\ -3 & \textit{mod } 21. \end{cases}$$

(Explain carefully how you got your result.)

The main point is that if x and y both satisfy the three congruences, then their difference is divisible by $19 \cdot 20 \cdot 21$. Since -3 satisfies the three congruences, the

smallest positive number that satisfies the congruences is then $-3 + 19 \cdot 20 \cdot 21$. (There is no need to compute this value, but if you did compute it you should have gotten 7977.)

To see the main point, let $d = x - y$, so that d is divisible by each of 19, 20 and 21. Since $\gcd(19, 20) = 1$, d is divisible by $19 \cdot 20$ (by the first lemma of last Thursday's lecture). You can see that $\gcd(19 \cdot 20, 21) = 1$ by using the second lemma of that lecture—which is the next problem on this test. Applying the first lemma again, one gets that d is divisible by $(19 \cdot 20) \cdot 21$, as required.

6. *Use Bézout's theorem to prove that if a is relatively prime both to b and to c, then a is relatively prime to bc. In symbols:*

$$\gcd(a, b) = \gcd(a, c) = 1 \quad \stackrel{?}{\longrightarrow} \quad \gcd(a, bc) = 1.$$

The statement to be proved is the second lemma in the notes for the February 14 lecture. You can find the proof there. For convenience:

"To prove the desired conclusion, it is enough to show that 1 is a linear combination of a and bc; indeed, if this is true, then any divisor of both a and bc will be a divisor of 1 and therefore equal to 1.

"To write 1 as a linear combination of a and bc, we use Bézout to write

$$1 = ax + by, \qquad 1 = za + wc.$$

Multiplying these together gives

$$1 = (ax + by)(za + wc) = a(xza + wcx + byz) + yw \cdot bc,$$

a linear combination of a and bc."