MATH 54: MIDTERM 2 SOlUTIONS

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1. Basis for a set of vectors

The specified set of vectors consists precisely of

$$\left\{ \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbf{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} \right\}.$$

The set of vectors on the right may not be linearly independent; to find a basis we should identify a linearly independent subset. This is equivalent to finding a basis for the column space of the matrix with these vectors as its columns. We make the following row reduction:

$$\begin{bmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \\ 0 & 7 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there are pivots in the first two columns, we conclude that a basis for the set of vectors given is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}}$$

2. Rank is unchanged by augmenting with an element of the column space

The rank of A is the number of pivots in its REF. Since $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$, the matrix $[A : \mathbf{b}]$ represents a consistent system and thus its REF has no pivot in the augmented column. Thus the number of pivots of the REF of A equals that of $[A : \mathbf{b}]$; in other words, they have the same rank.

3. Matrix power

The characteristic polynomial of this matrix is

$$\chi_A(t) = \det \begin{bmatrix} 4-t & -3 \\ 2 & -1-t \end{bmatrix} = (4-t)(-1-t) + 6 = t^2 - 3t + 2 = (t-1)(t-2).$$

Thus the eigenvalues of A are $\lambda = 1, 2$. The corresponding eigenspaces are

$$\ker(A - \mathrm{Id}) = \ker \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} = \operatorname{span}\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \} \qquad \text{and} \qquad \ker(A - 2\,\mathrm{Id}) = \ker \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} = \operatorname{span}\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \}.$$

If we form the matrix P whose columns are the eigenvectors, we may diagonalize $A = PDP^{-1}$:

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \implies A^{20} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1^{20} & \\ & 2^{20} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 2^{20} & -2^{20} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -2 + 3 \cdot 2^{20} & 3 - 3 \cdot 2^{20} \\ -2 + 2^{21} & 3 - 2^{21} \end{bmatrix}}$$

4. Lengths and distances

The norms of $\mathbf{u} = (3, 4, 3)$ and $\mathbf{v} = (2, -3, 2)$ are

$$\|\mathbf{u}\| = \sqrt{3^2 + 4^2 + 3^2} = \boxed{\sqrt{34}} \qquad \text{and} \qquad \|\mathbf{v}\| = \sqrt{2^2 + (-3)^2 + 2^2} = \boxed{\sqrt{17}}.$$

The distance between them is

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(3-2)^2 + (4-(-3))^2 + (3-2)^2} = \boxed{\sqrt{51}}.$$

And they are orthogonal since $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + 4 \cdot (-3) + 3 \cdot 2 = 6 - 12 + 6 = 0$.

5. Orthogonal basis

Since W is spanned by the three linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, it is three-dimensional. Therefore any three linearly independent vectors in W will form a basis for W. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are nonzero orthogonal vectors, they are linearly independent, and since there are three of them, they constitute a basis for W.