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Math54 Midterm II, Spring 2019

This is a closed everything exam, except a standard one-page cheat sheet (on one-side only). You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Write your personal information below.

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1. Find a basis for the set of all vectors of the form, with scalars α, β, γ:

$$\begin{pmatrix} \alpha - 2\beta + 5\gamma \\ 2\alpha + 5\beta - 8\gamma \\ 3\alpha + \beta + \gamma \end{pmatrix}.$$

$$\Rightarrow \alpha \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{v_1} + \beta \underbrace{\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}}_{v_2} + \gamma \underbrace{\begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix}}_{v_3}$$

$$\boxed{\text{Basis} = \text{span}\left\{ \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -8 \\ 1 \end{bmatrix} \right\}}$$

v_1 is not included in the basis because it is a linear combination of v_2 and v_3 where $\underline{v_1 = 2v_2 + v_3}$

2. Let A be an $m \times n$ matrix and $\mathbf{b}$ an m dimensional vector. Suppose that there is a solution $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$. Show that $\mathbf{rank}\,(A) = \mathbf{rank}\,([A,\ \mathbf{b}])$.

Since there is a solution to $Ax=b$, the columns of A are Linearly independent, meaning that there is a pivot in every column, and $\text{rank}(A) = \binom{\text{number of}}{\text{pivot columns}} = n$.

The augmented matrix is of dimensions $m \times (n+1)$.

Using the rank theorem, we have:

$$\text{rank}([A,b]) + \dim \text{Nul}([A,b]) = n+1$$

$$\text{rank}([A,b]) + 1 = n+1$$

$$\boxed{\text{rank}([A,b]) = n = \text{rank}(A)}$$

This is equal to 1 because augmenting A with b adds 1 non-pivot column, meaning that b is in the null space of A. Since there is now 1 vector in $\text{Nul}([A,b])$, $\dim \text{Nul}([A,b])$ is equal to 1.

3. Compute A^{20}, where $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$. $A = PDP^{-1}$

$$(A - \lambda I) = \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} \Rightarrow (4-\lambda)(-1-\lambda) + 6 = 0$$

$$-4 - 3\lambda + \lambda^2 + 6 = 0$$

$$2 - 3\lambda + \lambda^2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, \lambda = 2$$

for $\lambda = 1$: $\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = 2$: $\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \to \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ $P^{-1} \Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & -2 & -1 & 0 \end{array}\right] \to \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$A = PDP^{-1}$$

$$A^{20} = PD^{20}P^{-1}$$

$$\boxed{A^{20} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{20} \end{bmatrix} \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix}}$$

4. Let $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$. Find the lengths of these vectors, the distance between them. Are the two vectors orthogonal?

$$u \cdot v = 3(2) + 4(-3) + 3(2) = 6 - 12 + 6 = \boxed{0 \Rightarrow u \text{ and } v \text{ are orthogonal}}$$

$$\|u\| = \sqrt{u \cdot u} = \sqrt{3^2 + 4^2 + 3^2} = \sqrt{9 + 16 + 9} = \sqrt{34}$$

$$\boxed{\|u\| = \sqrt{34}}$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{2^2 + (-3)^2 + 2^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$$

$$\boxed{\|v\| = \sqrt{17}}$$

$$\text{dist}(u, v) = \|u - v\| = \sqrt{(3-2)^2 + (4+3)^2 + (3+2)^2}$$

$$= \sqrt{1^2 + 7^2 + 5^2}$$

$$= \sqrt{1 + 49 + 25}$$

$$\boxed{\text{dist}(u, v) = \|u - v\| = \sqrt{75}}$$

5. Let $W = \textbf{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ with linearly independent vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. Show that if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are non-zero orthogonal vectors in W, then $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are a basis for W.

For $1 \le k \le 3$, let $W_k = \text{span}(u_1, u_2, u_3)$. Set $v_1 = u_1$, so that $\text{Span}\{v_1\} = \text{span}\{u_1\}$. Suppose, for some $k < 3$, we have constructed $v_1 \dots v_3$ so that $\{v_1, v_2, v_3\}$ is an orthogonal basis for W_k. Hence, $\{v_1, v_2, v_3\}$ is an orthogonal set of non-zero vectors in the same dimensional space as W. By the Basis Therem, this set is an orthogonal basis for W.

we know this b/c u_1, u_2, u_3 are LID which means they cannot $= 0$, since all vectors are LD with $\{0\}$, so $v_1 = u_1$, $v_2 = u_2$, $v_3 = u_3$ are also non-zero.