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Math54 Midterm I, Spring 2019

This is a closed book exam. Everyone is allowed a one-page cheat-sheet but no calculators. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Your Name: Jack Smith

Your GSI: Tahsin Saffat

Your SID: 3032908127

1. Let $\mathbf{T} : \mathcal{R}^m \to \mathcal{R}^n$ be a linear transformation and suppose $T(\mathbf{u}) = \mathbf{v}$. Show that $T(-\mathbf{u}) = -\mathbf{v}$.

In order for $T: R^m \to R^n$ to be a linear transformation

↳ $T(cu) = cT(u)$ must be true

So if $T(v) = v$ then,

$T(-u) = -(T(u)) = -(v) = -v$

$T(-1 \cdot u) = -1(T(u)) = -1(v) = -v$

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$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 24 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 1 & 6 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1/3 \end{bmatrix}$

2. Determine h such that the solution set of the system

$$\begin{pmatrix} 1 & 3 \\ 4 & h \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

(i) is empty, (ii) contains a unique solution.

$\left(\begin{matrix} 4 & h \\ 1 & 3 \end{matrix}\middle| \begin{matrix} 8 \\ 1 \end{matrix}\right) \to \left(\begin{matrix} 1 & h/4 \\ 1 & 3 \end{matrix}\middle| \begin{matrix} 2 \\ 1 \end{matrix}\right)$

$\left(\begin{matrix} 1 & h/4 \\ 0 & 3-h/4 \end{matrix}\middle| \begin{matrix} 2 \\ -1 \end{matrix}\right)$ $3 - h/4 \neq 0$ $h \neq 12$

$\begin{matrix} ① \\ ② \end{matrix} \left(\begin{matrix} 1 & 3 \\ 4 & h \end{matrix}\middle| \begin{matrix} 1 \\ 8 \end{matrix}\right) \xrightarrow{1/4 ②} \left(\begin{matrix} 1 & 3 \\ 1 & h/4 \end{matrix}\middle| \begin{matrix} 1 \\ 2 \end{matrix}\right) \xrightarrow{②-①} \left(\begin{matrix} 1 & 3 \\ 0 & h/4 - 3 \end{matrix}\middle| \begin{matrix} 1 \\ 1 \end{matrix}\right) \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \end{bmatrix}$

i) $\frac{h}{4} - 3 = 0$, so $\frac{h}{4} = 3$,

↳ $\boxed{h = 12}$ This makes the aug. matrix inconsistent so no sol'n

ii) $\frac{h}{4} - 3 \neq 0 \to$ So $\boxed{h \neq 12}$ ~~$h \neq 24$~~

$\begin{bmatrix} 1 & 3 \\ 4 & h \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \to \left[\begin{matrix} 1 & 3 \\ 4 & h \end{matrix}\middle| \begin{matrix} 1 \\ 8 \end{matrix}\right]$ $1 - \frac{12}{h-12} \neq 0$

$\to \left[\begin{matrix} 1 & 3 \\ 0 & h-12 \end{matrix}\middle| \begin{matrix} 1 \\ 4 \end{matrix}\right] \to \left[\begin{matrix} 1 & 3 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 1 \\ 4/(h-12) \end{matrix}\right] \to \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 1 - 12/(h-12) \\ 4/(h-12) \end{matrix}\right]$

$h \neq 12$

i) inconsistent if $h = 12$

ii) unique sol'n if $h \neq 12$

~~$h \neq 24$~~

~~$1 \neq \frac{12}{h-12}$~~

~~$h - 12 \neq 12$~~

~~$h \neq 24$~~

3. Mark each statement TRUE or FALSE. Do not need to justify each answer.

(a) Let $A \in \mathcal{R}^{n \times n}$ be invertible. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b} \in \mathcal{R}^n$.

(b) Let $A \in \mathcal{R}^{n \times n}$ be invertible. Then the inverse of A^{-1} is A itself.

a) True

b) True

$(A^{-1})^{-1} = A$

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4. Compute the determinant of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix}.$$

$$\det(A) = 0\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$$

$$= 0(0-3) - 1(4-2) + 2(3-0)$$

$$= 0 - 1(2) + 2(3) = -2 + 6 = \boxed{4} \checkmark$$

$$\det(A) = 0\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 0 - 1(4-6) + 2(1-0)$$

$$= 0 - 1(-2) + 2 = 0 + 2 + 2 = \boxed{4} \checkmark$$

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5. Let $A, B \in \mathcal{R}^{n\times n}$ be square matrices.

(a) Assume that B is invertible. Show that $\det(B\,A\,B^{-1}) = \det(A)$.

(b) Assume that $A^T A = I$. Show that $\det(A) = \pm 1$.

a) B is invertible

↳ $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1})$

where $\det(B) \neq 0$

$= \det(B)\det(B^{-1})\det(A)$

$= \det(BB^{-1})\det(A)$

$= \det(I)\det(A)$

$= 1 \cdot \det(A)$

$\det(BAB^{-1}) = \det(A)$

Because if A & B are $n\times n$ and $AB = I$ then $A = B^{-1}$ and $B = A^{-1}$ and A and A^T are $n\times n$

b) $A^T A = I$

↳ A is square $A^T = A^{-1}$ & $A = (A^T)^{-1}$

$\det(A) = \det(A^T)$ is always true so $\det(A) = \det(A^{-1})$

Also $\det(A) = 1/\det(A^{-1})$

$\det(A)\det(A^{-1}) = 1$ and $\det(A) = \det(A^{-1})$

$(\det(A))^2 = 1$ $\det(A) = \sqrt{1} = \pm 1$