# MATH 1B MIDTERM 2 (001)

# PROFESSOR PAULIN

**DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.**

**CALCULATORS ARE NOT PERMITTED**

**THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM**

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x + \frac{k(k-1)(k-2)}{3!} + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

**This exam consists of 5 questions. Answer the questions in the spaces provided.**

1. Determine if the following series converge or diverge. If convergent you do not need to give the sum. Carefully justify your answers.

   (a) (10 points)

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+n+2}$$

**Solution:**

$\geq 0$

$$\lim_{n\to\infty} \frac{\left(\frac{n^2-1}{n^3+n+2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^3-n}{n^3+n+2} = 1 > 0$$

$\geq 0$

L.C.T.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2-1}{n^3+n+2} \quad \underline{\text{divergent}}$$

   (b) (15 points)

$$\sum_{n=1}^{\infty} n\tan\left(\frac{1}{n^3}\right)$$

**Solution:**

L'Hospital

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sec^2(x)}{1} = 1$$

$\geq 0$

$$\Rightarrow \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n\tan\left(\frac{1}{n^3}\right)}{\frac{1}{n^2}} = 1 > 0$$

$\geq 0$

L.C.T.

$$\Rightarrow \sum_{n=1}^{\infty} n\tan\left(\frac{1}{n^3}\right) \quad \underline{\text{convergent}}.$$

2. (25 points) Determine if the following series is convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$$

**Solution:**

$$a_n = \frac{n^n}{(2n)!} \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{(2n+1)(2n+2)}$$

$$= \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{(2n+1)\cdot 2}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$$

Ratio Test

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{(2n)!} \quad \underline{\text{convergent}}$$

3. (25 points) Using only the integral test, determine whether the following series is absolutely convergent, conditionally convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{\sqrt{\ln(n)}}{n}$$

**Solution:**

positive, cts on $[1,\infty)$

$$f(x) = \frac{\sqrt{\ln(x)}}{x} \Rightarrow f'(x) = \frac{\frac{1}{2}\ln(x)^{-\frac{1}{2}} \cdot \frac{1}{x} \cdot x - 1 \cdot \ln(x)^{\frac{1}{2}}}{x^2}$$

$$= \frac{\ln(x)^{-\frac{1}{2}}}{x^2}\left(\frac{1}{2} - \ln(x)\right)$$

$\Rightarrow f'(x) < 0$ for all $x > e^{1/2}$

$\Rightarrow f(x)$ decreasing on $(e^{1/2}, \infty)$ $\Rightarrow$ Can apply integral test

$$u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$$

$$\Rightarrow \int \frac{\sqrt{\ln(x)}}{x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(\ln(x))^{3/2} + C$$

$$\Rightarrow \int_1^{\infty} \frac{\sqrt{\ln(x)}}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\sqrt{\ln(x)}}{x}\,dx = \lim_{t\to\infty} \frac{2}{3}(\ln(t))^{3/2}$$

$$= \infty$$

Integral Test.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{\ln(n)}}{n} \quad \underline{\text{divergent}}.$$

4. (25 points) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n}}{(4+1/n)^n}.$$

**Solution:**

$$a_n = (-1)^n \frac{x^{2n}}{(4+\frac{1}{n})^n} \Rightarrow \sqrt[n]{|a_n|} = \frac{|x|^2}{4+\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|x|^2}{4}$$

Root Test

$\Leftrightarrow |x| < 2$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(4+\frac{1}{n})^n} \text{ abs. conv. if } \frac{|x|^2}{4} < 1$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(4+\frac{1}{n})^n} \text{ div. if } \frac{|x|^2}{4} > 1$$

$\Leftrightarrow |x| > 2$

$\Rightarrow$ Radius of Convergence $= 2$

5. (a) (20 points) Calculate the Taylor series (centered at $x = -1$) of the function

$$f(x) = (x^2 + 2x + 1)\sin(x + 1)$$

Make sure to write a general term of the series.

**Solution:**

$(-\infty, \infty)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

$$\Rightarrow \sin(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x+1)^{2n-1}}{(2n-1)!}$$

$(-\infty, \infty)$ $(-\infty, \infty)$

$$\Rightarrow (x^2 + 2x + 1)\sin(x+1) = (x+1)^2 \sin(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x+1)^{2n+1}}{(2n-1)!}$$

Taylor series at $x = -1$

(b) (5 points) Using part (a), or otherwise, calculate $f^{(2n+1)}(-1)$, where $n$ is a positive whole number. You do not need to simplify your answer.

**Solution:**

$$C_{2n+1} = \frac{(-1)^{n-1}}{(2n-1)!} = \frac{f^{(2n+1)}(-1)}{(2n+1)!}$$

$$\Rightarrow f^{(2n+1)}(-1) = (-1)^{n-1} \frac{(2n+1)!}{(2n-1)!} = (-1)^n (2n)(2n+1)$$

END OF EXAM