MATH 1B MIDTERM 1 (001)

PROFESSOR PAULIN

> ### DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

> ### CALCULATORS ARE NOT PERMITTED

> ### THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Formulae

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

Name: _____

Student ID: _____

GSI's name: _____

1. Compute the following integrals:

   (a) (10 points)

   $$\int xe^{2x}\ dx$$

   Solution:

$$f(x) = x\ ,\quad g'(x) = e^{2x}$$
$$f'(x) = 1\ ,\quad g(x) = \tfrac{1}{2}e^{2x}$$

$$\Rightarrow \int 2e^{2x}\ dx = \tfrac{x}{2}e^{2x} - \tfrac{1}{2}\int e^{2x}\ dx$$

$$= \tfrac{x}{2}e^{2x} - \tfrac{1}{4}e^{2x} + C$$

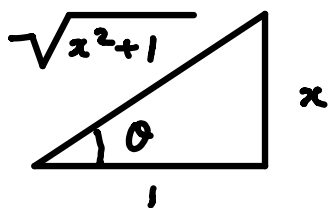   (b) (15 points)

   $$\int \frac{1}{x^2\sqrt{x^2+1}}\ dx$$

   Solution:

$$x = \tan\theta \implies \frac{dx}{d\theta} = \sec^2\theta \implies dx = \sec^2\theta\ d\theta$$

$$\Rightarrow \int \frac{1}{x^2\sqrt{x^2+1}}\ dx = \int \frac{1}{\tan^2\theta \sec\theta}\cdot\sec^2\theta\ d\theta = \int \sin^{-2}\theta \cos\theta\ d\theta$$

$$u = \sin\theta \implies \frac{du}{d\theta} = \cos\theta \implies d\theta = \frac{du}{\cos\theta}$$

$$\Rightarrow \int \sin^{-2}\theta \cos\theta\ d\theta = \int u^{-2}\ du = \frac{-1}{u} + C = \frac{-1}{\sin\theta} + C$$

$$= \frac{-\sqrt{x^2+1}}{x} + C$$

2. (a) (15 points) Express the following rational function

$$\frac{x^3 + x^2 + 3x + 1}{x^4 + 2x^2 + 1}$$

as a sum of partial fractions.

Solution:

$$\frac{x^3 + x^2 + 3x + 1}{x^4 + 2x^2 + 1} = \frac{x^3 + x^2 + 3x + 1}{(x^2+1)^2} = \frac{Ax + B}{x^2+1} + \frac{Cx + D}{(x^2+1)^2}$$

$$= \frac{(Ax + B)(x^2+1) + Cx + D}{(x^2+1)^2}$$

$$= \frac{Ax^3 + Bx^2 + (A+C)x + (B+D)}{(x^2+1)^2}$$

$$A = 1$$
$$\Rightarrow \quad B = 1$$
$$A + C = 3$$
$$B + D = 1$$

$$\Rightarrow \quad \begin{array}{l} A = 1 \\ B = 1 \\ C = 2 \\ D = 0 \end{array} \quad \Rightarrow \quad \frac{x^3 + x^2 + 3x + 1}{x^4 + 2x^2 + 1} = \frac{x+1}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

(b) (10 points) Hence evaluate the integral

$$\int \frac{x^3 + x^2 + 3x + 1}{x^4 + 2x^2 + 1}\, dx$$

Solution:

$$\Rightarrow \quad \int \frac{x^3 + x^2 + 3x + 1}{x^4 + 2x^2 + 1}\, dx = \int \frac{x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx$$

$$u = x^2 + 1$$
sub.

$$+ \int \frac{2x}{(x^2+1)^2}\, dx$$

$$= \frac{1}{2} \ln|x^2 + 1| + \arctan(x) - \frac{1}{x^2+1} + C$$

3. (25 points) Find the arc length of the curve

$$y = \frac{x^4}{16} + \frac{1}{2x^2}$$

between $x = 1$ and $x = 2$.

Solution:

$$f(x) = \frac{x^4}{16} + \frac{1}{2x^2} \implies f'(x) = \frac{x^3}{4} - \frac{1}{x^3}$$

$$\implies f'(x) = \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}$$

$$\implies 1 + f'(x) = \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} = \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2$$

$$\implies \text{Arc Length} = \int_1^2 \frac{x^3}{4} + \frac{1}{x^3} \, dx$$

$$= \frac{x^4}{16} - \frac{1}{2x^2} \bigg|_1^2$$

$$= \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right)$$

$$= \frac{21}{16}$$

4. Determine if the following improper integrals are convergent or divergent. Justify your answers.

    (a) (10 points)
    $$\int_0^\infty \frac{\sin(x)+3}{2^x}dx$$

Solution:

$0 \le \sin(x) + 3 \le 4 \quad \Rightarrow \quad 0 \le \frac{\sin(x)+3}{2^x} \le \frac{4}{2^x}$

$\int_0^\infty \frac{4}{2^x}dx = \lim_{t\to\infty} \int_0^t \frac{4}{2^x} = \lim_{t\to\infty} \frac{4}{\ln(\frac{1}{2})2^x}\Big|_0^t$

$= \lim_{t\to\infty} \frac{4}{\ln(\frac{1}{2})2^t} - \frac{4}{\ln(\frac{1}{2})} = \frac{-4}{\ln(\frac{1}{2})} \quad \leftarrow$ **convergent**

Comparison Test $\Rightarrow \int_0^\infty \frac{\sin(x)+3}{2^x}dx$ convergent

    (b) (15 points)
    $$\int_{-1}^2 \frac{x^2}{x^3-1}dx$$

$\frac{x^2}{x^3-1}$ has vertical asymptote at $x=1$

$u = x^3 - 1 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow \int \frac{x^2}{x^3-1}dx = \int \frac{1}{3}\cdot\frac{1}{u}du$

$= \frac{1}{3}\ln|x^3-1| + C$

$\Rightarrow \int_1^2 \frac{x^2}{x^3-1}dx = \lim_{t\to1^+} \int_t^2 \frac{x^2}{x^3-1}dx$

$\lim_{t\to1^+} t^3 - 1 = 0$

$\lim_{v\to0} \ln|v| = -\infty$

$= \lim_{t\to1^+} \frac{1}{3}\ln(7) - \frac{1}{3}\ln|t^3-1|$

$= \infty \quad \Rightarrow \int_{-1}^2 \frac{x^2}{x^3-1}dx$ **div**

5. (a) (15 points) For n a positive integer, let T_n be the trapezoidal approximation of the definite integral

$$\int_0^{\pi/4} 1 - \tan(x)\, dx.$$

How large do we need to choose n to be to guarantee that the estimate is within 0.01 of the true value? You do not need to give an exact answer, just a carefully justified bound.

Solution:
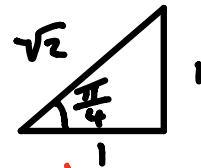
$$f(x) = 1 - \tan(x) \implies f'(x) = -\sec^2(x)$$

$$\implies f''(x) = -2\sec^2(x)\tan(x)$$

tan, sec increasing on $[0, \frac{\pi}{4}]$

$$|f''(x)| = 2|\sec(x)|^2 \cdot |\tan(x)| \le 2 \cdot (\sqrt{2})^2 \cdot 1 = 4$$

$$\implies |E_{T_n}| \le \frac{4 \cdot (\frac{\pi}{4} - 0)^3}{12\, n^2}$$

Need n such that

$$\frac{4 \cdot (\frac{\pi}{4} - 0)^3}{12\, n^2} \le 0.01 \iff n \ge \sqrt[2]{\frac{4 \cdot (\frac{\pi}{4})^3}{12 \cdot (0.01)}}$$

$n > 0$

(b) (10 points) Is this approximation an overestimate or an underestimate? Be sure to justify your answer.

Solution:

$$f''(x) = -2\sec^2(x)\tan(x) < 0 \quad \text{on} \quad (0, \frac{\pi}{4})$$

$$\implies y = f(x) \text{ concave down on } (0, \frac{\pi}{4})$$

$$\implies T_n \text{ is an underestimate}$$

END OF EXAM