1. Consider the problem:

$$f(x)'' + 7f(x) = -2 \qquad x \in (0,b) \tag{1}$$
$$f'(0) = c \tag{2}$$
$$f(b) = 0 \tag{3}$$

(a) (5pts) Identify the essential and the natural boundary conditions.

(b) (5pts) Define the space of trial solutions, $\mathcal{S}$.

(c) (5pts) Define the space of admissible test functions, $\mathcal{V}$.

(d) (10pts) The weak form problem statement can be written as: Find $f \in \mathcal{S}$ such that $a(w,f) = l(w)$ for all $w \in \mathcal{V}$. Show that a and l are given by

$$a(w,f) = \int_0^b w'f' - w7f\,dx \tag{4}$$
$$l(w) = -w(0)\,c + \int_0^b 2w\,dx \tag{5}$$

(e) (10pts) Considering shape function expansions of the form

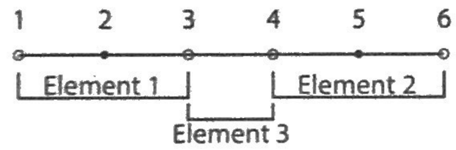
$$f(x) = \boldsymbol{N}(x)\boldsymbol{f} \tag{6}$$
$$f'(x) = \boldsymbol{B}(x)\boldsymbol{f}\,, \tag{7}$$

etc., where $\boldsymbol{N}$ and $\boldsymbol{B}$ can be assumed to be given, find the governing finite element matrix equations $\boldsymbol{K}\boldsymbol{f} = \boldsymbol{F}$ to this problem.

2. Consider the strong form equation for a one-dimensional elastic bar

$$AEu'' + b = 0\,, \tag{8}$$

where AE is a constant and $b(x)$ is given. The domain of interest has been discretized by 2 quadratic (Lagrange) elements and 1 linear (Lagrange) element as shown.



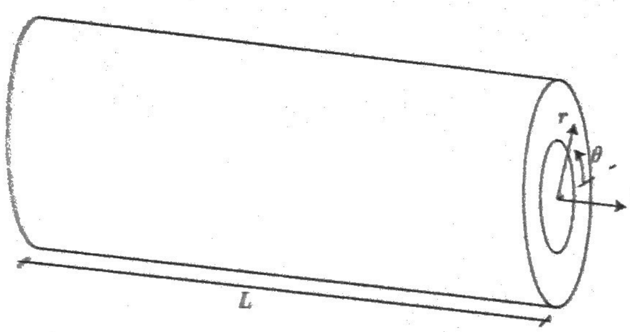
Note the nodal spacing is uniform, with spacing h. The element stiffness matrices for the linear and the quadratic elements are given by

$$\boldsymbol{k}^e = \frac{AE}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \boldsymbol{k}^e = \frac{AE}{h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\,. \tag{9}$$

(a) (10pts) What is the size of the global stiffness matrix?

(b) (15pts) Write out an expression for the global stiffness matrix.

3. Consider heat conduction in a circular cylinder with insulated ends and prescribed temperature on the inner and outer lateral faces that is independent of z, the axial coordinate. Further assume that the heat supply is not a function of z – i.e. $Q(r, \theta)$.



In this setting the temperature is only a function of radial position, r, and polar angle θ and the temperature $T(r, \theta)$ can be determined by solving a two-dimensional problem in the r, θ-plane that is governed by the weak form equation

$$L \int_A (\nabla w)^T D \nabla T \, r dr d\theta = L \int_A w Q \, r dr d\theta \tag{10}$$

where the gradient in cylindrical coordinates is given by

$$\nabla f = \frac{\partial f}{\partial r} \boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta} \boldsymbol{e}_\theta + \frac{\partial f}{\partial z} \boldsymbol{e}_z . \tag{11}$$

In matrix form, the weak form is given by

$$\boldsymbol{K}\boldsymbol{T} = \boldsymbol{F} \tag{12}$$

where $\boldsymbol{K}$ can be expressed in terms of the $\boldsymbol{B}$ matrix and $\boldsymbol{F}$ can be expressed in term of the $\boldsymbol{N}$ matrix.

(a) (10pts) The $\boldsymbol{B} = [\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots]$ matrix is defined in terms of the nodal B-matrices $\boldsymbol{B}_a$. What is the expression for $\boldsymbol{B}_a$ in this setting assuming a given set of shape functions N_a?

(b) (10pts) What is the expression for $\boldsymbol{K}$ in terms of the $\boldsymbol{B}$ matrix?

4

4. Consider the boundary value problem

$$u'' + Pu = 0 \tag{13}$$
$$u(0) = 0 \tag{14}$$
$$u(L) = 0\,, \tag{15}$$

where P is a fixed parameter. This boundary value problem always possesses the trivial solution $u(x) = 0$ for all values of P. However, for special values of P one can find non-trivial solutions, i.e. solutions where $u(x) \neq 0$.

(20pts) Use 2 linear elements to discretize the domain $[0, L]$ and estimate one such special value of P.

Hints:

(a) For linear elements one has the result that

$$\int_{\Omega_e} N_i^e N_j^e \, dx = \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{16}$$

where h is the size of the element.

(b) Your answer will be approximately 20% larger than the exact value of π^2/L^2.