Second Midterm Examination
Tuesday October 25 2016
6:05pm-7:05pm in 150 NorthGate Hall
Closed Books and Closed Notes

Question 1
A System of Two Particles
20 Points

A simple model for a toy consists of a pair of mass particles connected by a flexible element. As a preliminary to developing a model for the toy, consider a particle of mass m_1 which is free to move along a smooth plane curve (i.e., $y = f(x), z = 0$) and is connected by a linear spring of stiffness K and unstretched length ℓ_0 to a particle of mass m_2. Both particles are under the influence of the respective gravitational forces $-m_1 g\mathbf{E}_2$ and $-m_2 g\mathbf{E}_2$ and m_2 is free to move in space as shown in Figure 1.

$q_1 := \eta$
$q_2 := z$

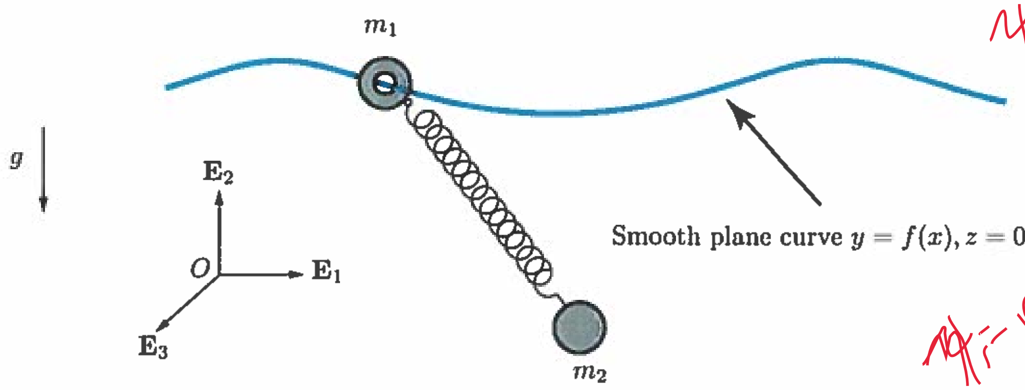


Figure 1: *A system of two particles. A particle of mass m_2 is suspended by a spring from a bead of mass m_1. The bead is free to move along a smooth plane curve.*

A coordinate system $\{x, \eta = y - f(x), z\}$ is chosen to parameterize the motion of m_1 and a spherical polar coordinate system $\{R, \theta, \phi\}$ is chosen to parameterize $\mathbf{r}_2 - \mathbf{r}_1$:

$$\mathbf{r}_1 = x\mathbf{E}_1 + (\eta + f(x))\,\mathbf{E}_2 + z\mathbf{E}_3, \qquad \mathbf{r}_2 = \mathbf{r}_1 + R\mathbf{e}_R. \tag{1}$$

(a) *(6 Points)* Compute the 12 vectors $\frac{\partial \mathbf{r}_i}{\partial q^k}$ where $q^1 = x, q^2 = R, q^3 = \theta, q^4 = \phi, q^5 = \eta$, and $q^6 = z$.

(b) *(9 Points)* Show that the constrained Lagrangian of the system of particles is

$$\tilde{L} = \frac{m_1 + m_2}{2}\dot{x}^2\left(1 + f^{'}f^{'}\right) + \frac{m_2}{2}\left(\dot{R}^2 + R^2\sin^2(\phi)\,\dot{\theta}^2 + R^2\dot{\phi}^2\right) + m_2\,(???) - ???? \tag{2}$$

where $f^{'} = \frac{df}{dx}$. For full credit, supply the missing terms in (2). Depending on how you collect terms, there will be at least 5 missing terms overall.

(c) *(5 Points)* Give a prescription for the constraint forces $\mathbf{F}_{c_1}$ and $\mathbf{F}_{c_2}$ acting on the respective particles, and compute the following six summations:

$$\mathbf{F}_{c_1} \cdot \frac{\partial \mathbf{r}_1}{\partial q^k} + \mathbf{F}_{c_2} \cdot \frac{\partial \mathbf{r}_2}{\partial q^k}, \qquad k = 1, \ldots, 6. \tag{3}$$

Question 2

A Vibration Absorber

30 Points

Vibration absorbers are mounted on bridges, skyscrapers, cables and machines in order to reduce or eliminate unwanted vibration. One simple design of a vibration absorber consists of a mass-spring-dashpot which is attached to a vibrating mass. By properly tuning the stiffness and damping of the attached system, the motion of the vibrating mass can be reduced and eventually eliminated. A simple two-particle model for such a system is shown in Figure 2. The absorber mass is m_2 and both particles are assumed to have a single degree-of-freedom. The following coordinate system is used to parameterize the motion of the particles:

$$\mathbf{r}_1 = (x_1 + \ell_{0_1} + a_1)\,\mathbf{E}_1 + y_1\mathbf{E}_2 + z_1\mathbf{E}_3, \qquad \mathbf{r}_2 = \mathbf{r}_1 + (x_2 + \ell_{0_2} + a_2)\,\mathbf{E}_1 + y_2\mathbf{E}_2 + z_2\mathbf{E}_3, \quad (4)$$

where the constants a_1 and a_2 are chosen such that the system has a rest state (i.e., is in equilibrium) when $x_1 = 0$ and $x_2 = 0$:

$$a_1 = \frac{m_1 g + m_2 g}{K_1}, \qquad a_2 = \frac{m_2 g}{K_2}. \quad (5)$$

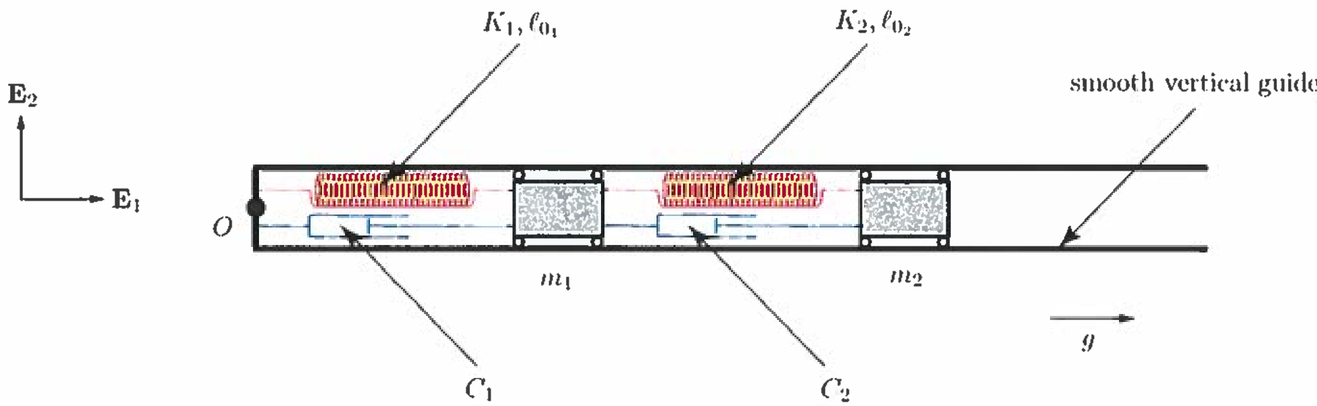


Figure 2: *A vibration absorption system modeled using a pair of particles. The springs are linear with unstretched lengths ℓ_{0_1} and ℓ_{0_2}, respectively, the dashpots are linear with constant positive damping coefficients ($C_1 > 0$ and $C_2 > 0$), and the motion of the system is unaffected by Coulomb friction.*

(a) *(4 Points)* Compute the four vectors $\frac{\partial \mathbf{r}_i}{\partial q^k}$ where $q^1 = x_1$ and $q^2 = x_2$.

(b) *(6 Points)* Establish expressions for $\tilde{T}$ and $\tilde{U}$ of the system of particles.

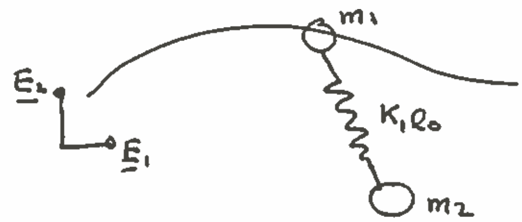
(c) *(9 Points)* What are the nonconservative forces $\mathbf{F}_{\mathrm{ncon}_1}$ and $\mathbf{F}_{\mathrm{ncon}_2}$ acting on the respective particles? Show that the combined power of these forces reduces the total energy of the system.

(d) *(9 Points)* Establish the equations of motion for the system of particles:

$$\begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (6)$$

(e) *(2 Points)* If a force $P_1 \cos(\omega_1 t)\,\mathbf{E}_1$ acts on m_1 and a force $P_2 \cos(\omega_2 t)\,\mathbf{E}_1$ acts on m_2, then how are the equation of motions (6) altered?

QUESTION 1



$$\underline{r}_1 = x\underline{E}_1 + (\eta + f)\underline{E}_2 + z\underline{E}_3$$

$$\underline{r}_2 = \underline{r}_1 + R\underline{e}_R$$

(a)

	q^1	q^2	q^3	q^4	q^5	q^6
	x	R	θ	ϕ	η	z
$\frac{\partial \underline{r}_1}{\partial q^K}$	$\underline{E}_1 + f'\underline{E}_2$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{E}_2$	$\underline{E}_3$
$\frac{\partial \underline{r}_2}{\partial q^K}$	$\underline{E}_1 + f'\underline{E}_2$	$\underline{e}_R$	$R\sin\phi\,\underline{e}_\theta$	$R\underline{e}_\phi$	$\underline{E}_2$	$\underline{E}_3$

(b)

$$\begin{aligned}
\tilde{T} = {} & \tfrac{1}{2} m_1 \; \dot{x}(\underline{E}_1 + f'\underline{E}_2) \cdot \dot{x}(\underline{E}_1 + f'\underline{E}_2) \\
& + \tfrac{1}{2} m_2 \; \dot{x}(\underline{E}_1 + f'\underline{E}_2) \cdot \dot{x}(\underline{E}_1 + f'\underline{E}_2) + \tfrac{1}{2} m_2 (\dot{R}^2 + R^2\dot{\theta}^2 \sin^2\phi + R^2\dot{\phi}^2) \\
& + m_2 \; \dot{x}(\underline{E}_1 + f'\underline{E}_2) \cdot (\dot{R}\underline{e}_R + R\dot{\phi}\underline{e}_\phi + R\dot{\theta}\sin\phi\,\underline{e}_\theta) \\
= {} & \tfrac{1}{2}(m_1 + m_2)\dot{x}^2(1 + f'f') + \tfrac{1}{2} m_2 (\dot{R}^2 + R^2\dot{\phi}^2 + R^2\dot{\theta}^2\sin^2\phi) \\
& + m_2 \big(\dot{x}\dot{R}(\cos\theta\sin\phi + f'\sin\theta\sin\phi) \\
& \qquad + \dot{x}R\dot{\phi}(\cos\theta\cos\phi + f'\sin\theta\cos\phi) + \dot{x}R\dot{\theta}\sin\phi(-\sin\theta + f'\cos\theta)\big)
\end{aligned}$$

$$\begin{aligned}
\tilde{u} = {} & m_1 g \, \underline{E}_2 \cdot \underline{r}_1 + m_2 g \, \underline{E}_2 \cdot \underline{r}_2 + \tfrac{1}{2} K (\|\underline{r}_2 - \underline{r}_1\| - \ell_0)^2 \\
= {} & (m_1 + m_2) g f + m_2 g R \sin\theta \sin\phi + \tfrac{1}{2} K (R - \ell_0)^2
\end{aligned}$$

$$\tilde{L} = \tilde{T} - \tilde{u}$$

(c)

$$\underline{F}_{c1} = \lambda_1 \nabla \eta + \lambda_2 \nabla z = \lambda_1(\underline{E}_2 - f'\underline{E}_1) + \lambda_2 \underline{E}_3$$

$$\underline{F}_{c2} = \underline{0}.$$

$$\begin{aligned}
\underline{F}_{c1} \cdot \frac{\partial \underline{r}_1}{\partial q^K} + \underline{F}_{c2} \cdot \frac{\partial \underline{r}_2}{\partial q^K} &= 0 \quad \text{for } K = 1, \ldots 4 \\
&= \lambda_1 \quad \text{for } K = 5 \\
&= \lambda_2 \quad \text{for } K = 6
\end{aligned}$$

$\Psi_1 := \eta$

$\Psi_2 := z$

$\pi_1 := \eta$

$\pi_2 := z$

$\boxed{\text{Comments on Question 1}}$

The particle m_1 is subject to 2 constraints

$$\eta = 0 \qquad z = 0$$

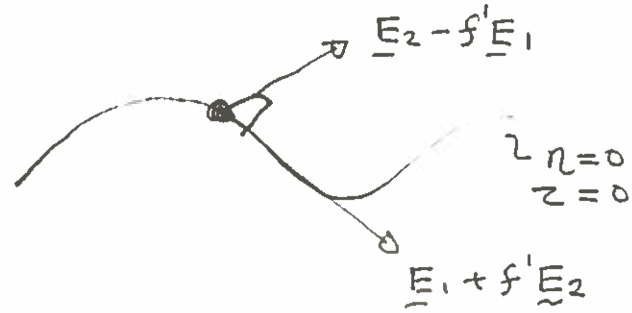
Hence, using the normality prescription / Lagrange prescription / specifying normal forces, we have

$$\underset{\sim}{F}_{c1} = \lambda_1 \nabla \eta + \lambda_2 \nabla z$$

$$= \lambda_1 (\underline{E}_2 - f' \underline{E}_1) + \lambda_2 \underline{E}_3 = \underset{\sim}{N}_1$$

Because m_2 is n't subject to a constraint

$$\underline{F}_{c2} = \underline{0}$$



Note that

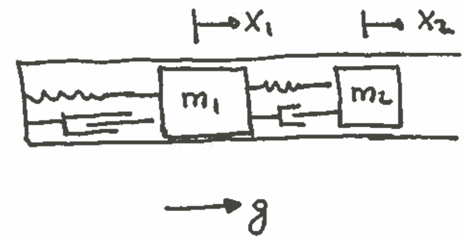
$$\underline{F}_{c1} \cdot \frac{\partial \underline{r}_1}{\partial q^K} + \underline{F}_{c2} \cdot \frac{\partial \underline{r}_2}{\partial q^K} = \underline{F}_{c1} \cdot \frac{\partial \underline{r}_1}{\partial q^K}$$

$$= 0 \quad \text{for} \quad K = 1, 2, 3, 4$$

$$= \lambda_1 \quad \text{for} \quad K = 5 \quad q^5 = \eta$$

$$= \lambda_2 \quad \text{for} \quad K = 6 \quad q^6 = z$$

$\boxed{\text{QUESTION 2}}$



$$\underline{r}_1 = (x_1 + \ell_{01} + a_1)\underline{E}_1 + y_1\underline{E}_2 + z_1\underline{E}_3$$

$$\underline{r}_2 = (x_2 + \ell_{02} + a_2)\underline{E}_1 + y_2\underline{E}_2 + z_2\underline{E}_3 + \underline{r}_1$$

$$a_1 = \frac{(m_1+m_2)g}{K_1} \qquad a_2 = \frac{m_2 g}{K_2}$$

(a) $\dfrac{\partial \underline{r}_1}{\partial x_1} = \underline{E}_1 \qquad \dfrac{\partial \underline{r}_1}{\partial x_2} = \underline{0} \qquad \dfrac{\partial \underline{r}_2}{\partial x_1} = \underline{E}_1 \qquad \dfrac{\partial \underline{r}_2}{\partial x_2} = \underline{E}_1$

(b) $\tilde{\underline{v}}_1 = \dot{x}_1\underline{E}_1 \qquad \tilde{\underline{v}}_2 = (\dot{x}_2 + \dot{x}_1)\underline{E}_1$

$$\tilde{T} = \tfrac{1}{2}m_1\tilde{\underline{v}}_1\cdot\tilde{\underline{v}}_1 + \tfrac{1}{2}m_2\tilde{\underline{v}}_2\cdot\tilde{\underline{v}}_2 = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2(\dot{x}_1+\dot{x}_2)^2$$

$$\tilde{u} = -(m_1 g x_1 + m_2 g(x_2 + x_1)) + \text{Constants}$$
$$+ \tfrac{1}{2}K_1(x_1+a_1)^2 + \tfrac{1}{2}K_2(x_2+a_2)^2$$

(c) $\underset{\sim}{F}_{ncon_1} = \underbrace{\lambda_1\underline{E}_2 + \lambda_2\underline{E}_3}_{\substack{=\underline{N}_1 \\ = \text{normal force on } m_1}} - c_1\dot{x}_1\underline{E}_1 - c_2(-\dot{x}_2)\underline{E}_2$

$\underset{\sim}{F}_{ncon_2} = \underbrace{\lambda_3\underline{E}_2 + \lambda_4\underline{E}_3}_{\substack{=\underline{N}_2 \\ = \text{Normal force on } m_2}} - c_2\dot{x}_2\underline{E}_1$

$$\dot{E} = \underline{F}_{ncon_1}\cdot\tilde{\underline{v}}_1 + \underline{F}_{ncon_2}\cdot\tilde{\underline{v}}_2$$
$$= \underline{N}_1\cdot\tilde{\underline{v}}_1 - c_1\dot{x}_1^2 - c_2(-\dot{x}_2)\dot{x}_1 + \underline{N}_2\cdot\tilde{\underline{v}}_2$$
$$-c_2\dot{x}_2(\dot{x}_1+\dot{x}_2)$$
$$= -c_1\dot{x}_1^2 - c_2\dot{x}_2^2 \le 0$$

so dampers drain energy from system.

(d) LEM

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^\alpha}\right) - \frac{\partial \tilde{\mathcal{L}}}{\partial q^\alpha} = \underline{F}_{ncon_1} \cdot \frac{\partial \underline{r}_1}{\partial q^\alpha} + \underline{F}_{ncon_2} \cdot \frac{\partial \underline{r}_2}{\partial q^\alpha} \qquad \begin{array}{l} q^1 = x_1 \\ q^2 = x_2 \end{array}$$

$$\tilde{\mathcal{L}} = \tilde{T} - \tilde{U}$$

Hence

$$\frac{d}{dt}\Big(m_1 \dot{x}_1 + m_2(\dot{x}_1 + \dot{x}_2)\Big) - \Big((m_1 g + m_2 g) - K_1(x_1 + a_1)\Big)$$

$$= \underline{F}_{ncon_1} \cdot \underline{E}_1 + \underline{F}_{ncon_2} \cdot \underline{E}_1$$

$$= -c_1 \dot{x}_1 + c_2 \dot{x}_2 - c_2 \dot{x}_2$$

$$= -c_1 \dot{x}_1$$

Note that $\quad m_1 g + m_2 g = K_1 a_1$

$$\frac{d}{dt}\Big(m_2(\dot{x}_2 + \dot{x}_1)\Big) - \Big(m_2 g - K_2(x_2 + a_2)\Big)$$

$$= \underline{F}_{ncon_2} \cdot \underline{0} + \underline{F}_{ncon_2} \cdot \underline{E}_1$$

$$= -c_2 \dot{x}_2$$

Note that $\quad m_2 g = K_2 a_2$

Hence

$$\begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(e) If $P_1 \cos \omega t\, \underline{E}_1$ acts on m_1 and $P_2 \cos \omega_2 t\, \underline{E}_2$ acts on m_2 then the rhs of the equations of motion change from

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} P_1 \cos \omega t + P_2 \cos \omega_2 t \\ P_2 \cos \omega_2 t \end{bmatrix}$$

Comments on Question 2

(b)

$$\tilde{u} = -m_1 g(x_1 + l_{01} + a_1) - m_2 g(x_1 + x_2 + l_{01} + l_{02} + a_1 + a_2) + \tfrac{1}{2}K_1(x_1 + a_1)^2 + \tfrac{1}{2}K_2(x_2 + a_2)^2 \qquad (*)$$

Substituting for $a_1 = \dfrac{m_1 g + m_2 g}{K_1}$ $\quad a_2 = \dfrac{m_2 g}{K_2}$

into

$$\tfrac{1}{2}K_1(x_1+a_1)^2 + \tfrac{1}{2}K_2(x_2+a_2)^2$$

$$= \tfrac{1}{2}K_1x_1^2 + \tfrac{1}{2}K_2x_2^2 + K_1x_1a_1 + K_2x_2a_2 + \tfrac{1}{2}K_1a_1^2 + \tfrac{1}{2}K_2a_2^2$$

$$= \tfrac{1}{2}K_1x_1^2 + \tfrac{1}{2}K_2x_2^2 + (m_1g + m_2g)x_1 + m_2g\,x_2 + \tfrac{1}{2}K_1a_1^2 + \tfrac{1}{2}K_2a_2^2$$

Hence

$$\tilde{u} = \tfrac{1}{2}K_1x_1^2 + \tfrac{1}{2}K_2x_2^2 + \left\{\tfrac{1}{2}K_1a_1^2 + \tfrac{1}{2}K_2a_2^2 - m_1g(l_{01}+a_1) - m_2g(l_{02}+l_{01}+a_1+a_2)\right\}$$

$$= \text{constant} \sim \text{say } \tilde{u}_0$$

$$\tilde{u} = \tfrac{1}{2}K_1x_1^2 + \tfrac{1}{2}K_2x_2^2 + \tilde{u}_0 \qquad (**)$$

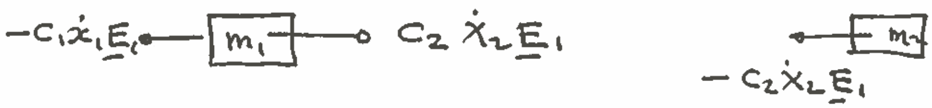
$\tilde{u}_0$ is a constant and so it won't effect E.E.M.

One can use $(*)$ or $(**)$ to arrive at the equations of motion but an explanation for why $(*)$ and $(**)$ was required in your solution if you use $(**)$ to get the equations of motion.

(c) and (d)

The damping forces on the particles are

$$\underline{F}_{d_1} = -c_1 \dot{x}_1 \underline{E}_1 + c_2 \dot{x}_2 \underline{E}_1$$
$$\underline{F}_{d_2} = -c_2 \dot{x}_2 \underline{E}_1$$



If you didn't know these expressions then you could reverse engineer them from the equations of motion. To do this assume

$$\underline{F}_{d_1} = \alpha \underline{E}_1 \qquad \underline{F}_{d_2} = \beta \underline{E}_1$$

Then as
$$\underline{F}_{ncon_1} = \underline{F}_{d_1} + \lambda_1 \underline{E}_2 + \lambda_2 \underline{E}_3$$
$$\underline{F}_{ncon_2} = \underline{F}_{d_2} + \lambda_3 \underline{E}_2 + \lambda_4 \underline{E}_3$$

$$\underline{F}_{ncon_1} \cdot \frac{\partial \underline{r}_1}{\partial x_1} + \underline{F}_{ncon_2} \cdot \frac{\partial \underline{r}_2}{\partial x_1} = \left(\underline{F}_{ncon_1} + \underline{F}_{ncon_2}\right) \cdot \underline{E}_1$$
$$= \alpha + \beta$$

$$\underline{F}_{ncon_1} \cdot \frac{\partial \underline{r}_1}{\partial x_2} + \underline{F}_{ncon_2} \cdot \frac{\partial \underline{r}_2}{\partial x_2} = \left(\underline{F}_{ncon_1}\right) \cdot \underline{0} + \underline{F}_{ncon_2} \cdot \underline{E}_1$$
$$= \beta$$

Hence looking at equations of motion

$$\alpha + \beta = -c_1 \dot{x}_1 \qquad \beta = -c_2 \dot{x}_2$$

So

$$\alpha = -c_1 \dot{x}_1 + c_2 \dot{x}_2$$

and

$$\underline{F}_{d_1} = \left(-c_1 \dot{x}_1 + c_2 \dot{x}_2\right) \underline{E}_1$$
$$\underline{F}_{d_2} = -c_2 \dot{x}_2 \underline{E}_1$$

as expected.