# First Midterm Examination
# Thursday September 24 2015
# Closed Books and Closed Notes
# Answer Both Questions

## Question 1
*A Kinetic Sculpture*
*20 Points*

As shown in Figure 1, a particle of mass $m$ is attached to a fixed point $O$ by an inextensible massless string whose length $\ell$ is varied as a function of time by a force $\mathbf{P}$. In addition to the tension force, a vertical gravitational force $-mg\mathbf{E}_3$ acts on the particle.

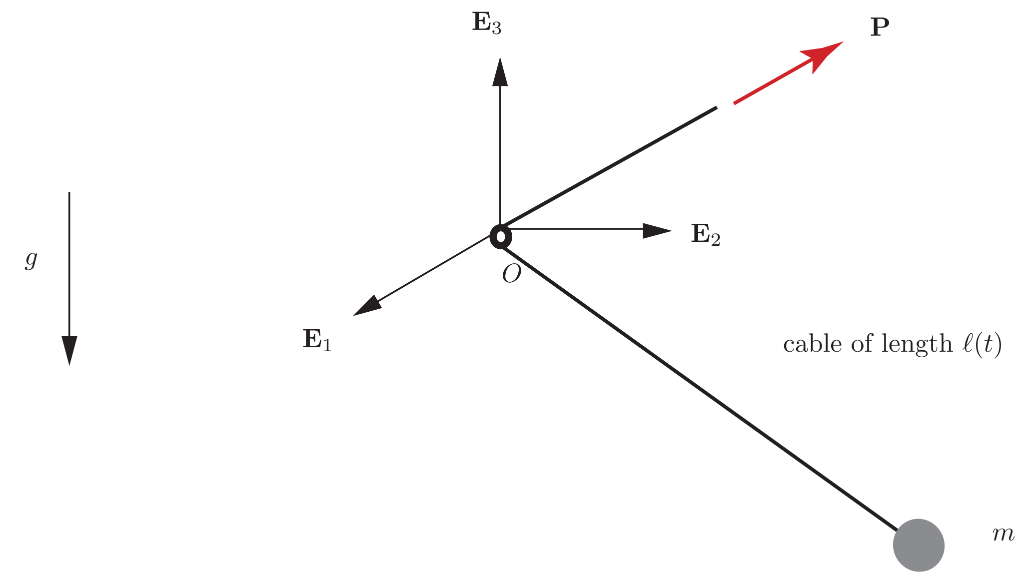



Figure 1: *Schematic of a particle of mass $m$ which is attached to a fixed point $O$ by a cable of length $\ell(t)$. A vertical gravitational force $-mg\mathbf{E}_3$ acts on the particle.*

In your answers to the questions below, please make use of the results on spherical polar coordinates on Page 3.

**(a)** *(5 Points)* What is the constraint on the motion of the particle? Give a prescription and physical interpretation for the constraint force enforcing this constraint.

**(b)** *(5 Points)* Establish the pair of second-order differential equations governing the motion of the particle.

**(c)** *(5 Points)* Show that, while the total energy $E$ of the particle isn't conserved, the angular momentum $\mathbf{H}_O \cdot \mathbf{E}_3$ of the particle is conserved during the motion of the particle.

**(d)** *(5 Points)* Show that the magnitude of the force $\mathbf{P}$ is

$$\|\mathbf{P}\| = \left| m\ddot{\ell} - m\ell\left(\dot{\theta}^2 \sin^2(\phi) + \dot{\phi}^2\right) + mg\cos(\phi) \right|. \tag{1}$$

## Question 2

*A Particle on a Catenary*

*30 Points*

As shown in Figure 2, a particle of mass $m$ is free to move on a rough catenary $y = A\cosh\left(\frac{x-x_0}{\ell}\right) + y_0$ and $z = 0$ where $A$, $\ell$, $y_0$, and $x_0$ are constants. The coefficients of static and dynamic friction between the particle and the catenary are denoted by $\mu_s$ and $\mu_k$, respectively.

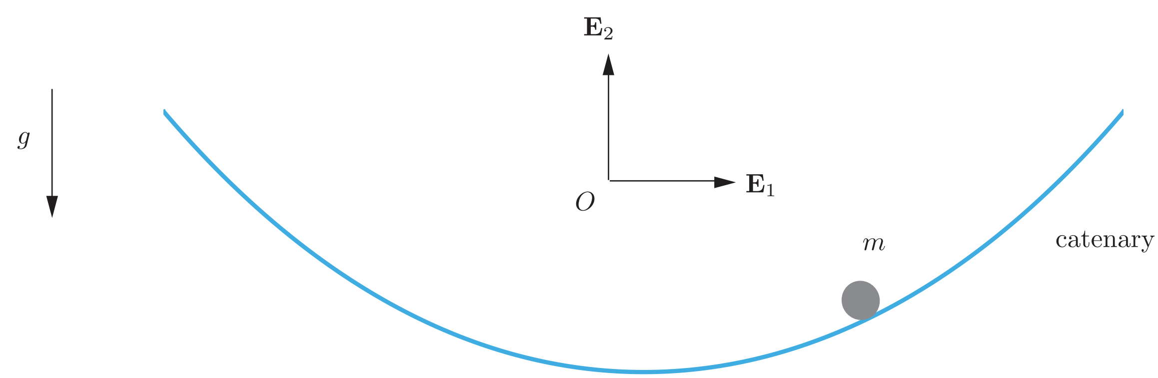



Figure 2: *Schematic of a particle of mass $m$ which is moving on a rough catenary in $\mathbb{E}^3$ under the influence of a gravitational force $-mg\mathbf{E}_2$.*

To establish the equations of motion for the particle, the following curvilinear coordinate system is defined for $\mathbb{E}^3$:

$$q^1 = x, \qquad q^2 = \eta = y - A\cosh\left(\frac{x-x_0}{\ell}\right), \qquad q^3 = z. \tag{2}$$

(**a**) *(5 Points)* What are the covariant basis vectors $\mathbf{a}_k$ for this coordinate system? You will find it helpful here and in the sequel to use the abbreviations $f = A\cosh\left(\frac{x-x_0}{\ell}\right)$ and $f_x = \frac{A}{\ell}\sinh\left(\frac{x-x_0}{\ell}\right)$

(**b**) *(5 Points)* Show that the contravariant basis vectors for this system are

$$\mathbf{a}^1 = \mathbf{E}_1, \qquad \mathbf{a}^2 = \mathbf{E}_2 - \frac{A}{\ell}\sinh\left(\frac{x-x_0}{\ell}\right)\mathbf{E}_1, \qquad \mathbf{a}^3 = \mathbf{E}_3. \tag{3}$$

Compute the matrix $\left[a^{ik}\right]$.

(**c**) *(15 Points)* Assuming the particle is in motion on the rough catenary $y = A\cosh\left(\frac{x-x_0}{\ell}\right) + y_0$ and $z = 0$ under a gravitational force $-mg\mathbf{E}_2$, establish the equations of motion for the particle. In your solution, give a clear prescription for the constraint force.

(**d**) *(5 Points)* Suppose that the particle is stationary on the rough curve. Determine the friction and normal forces acting on the particle.

**Notes on Spherical Polar Coordinates**

Recall that the spherical polar coordinates $\{R, \phi, \theta\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}\,, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right)\,, \quad \phi = \arctan\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right).$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

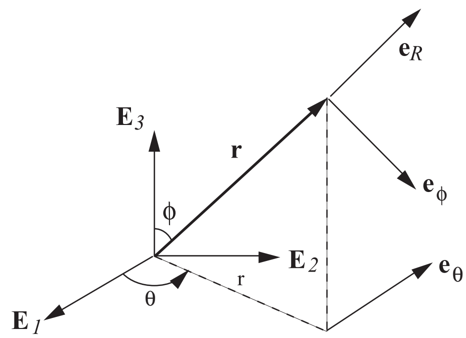



Figure 3: Spherical polar coordinates

For the coordinate system $\{R, \phi, \theta\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_R\,, \quad \mathbf{a}_2 = R\mathbf{e}_\phi\,, \quad \mathbf{a}_3 = R\sin(\phi)\mathbf{e}_\theta.$$

In addition, the contravariant basis vectors are

$$\mathbf{a}^1 = \mathbf{e}_R\,, \quad \mathbf{a}^2 = \frac{1}{R}\mathbf{e}_\phi\,, \quad \mathbf{a}^3 = \frac{1}{R\sin(\phi)}\mathbf{e}_\theta.$$

For a particle of mass $m$ which is unconstrained, the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_O$ and kinetic energy $T$ of the particle are

$$\begin{aligned} \mathbf{G} &= m\dot{R}\mathbf{a}_1 + m\dot{\phi}\mathbf{a}_2 + m\dot{\theta}\mathbf{a}_3\,, \\ \mathbf{H}_O &= mR^2\left(\dot{\phi}\mathbf{e}_\theta - \dot{\theta}\sin(\phi)\mathbf{e}_\phi\right)\,, \\ T &= \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2(\phi)\dot{\theta}^2\right). \end{aligned}$$

In grading this midterm exam, I noticed that the following errors frequently appeared:

1. For Problem 1, while $\|\mathbf{P}\| = \|\mathbf{F}_c = \lambda\mathbf{e}_R\|$, this doesn't imply that $\mathbf{P} = \mathbf{F}_c$. Also some student erroneously wrote that $\mathbf{F} = \mathbf{F}_c + \mathbf{P} - mg\mathbf{E}_3$. This is not the case. The correct statement is $\mathbf{F} = \mathbf{F}_c - mg\mathbf{E}_3 = \lambda\mathbf{e}_R - mg\mathbf{E}_3$. The constraint force acting on the particle is equivalent to a tension force $\lambda\mathbf{e}_R$.
2. For Problem 2 (c), many students incorrectly wrote down an expression for $\mathbf{F}_c$. This force consists of a normal force and a dynamic friction force. Note that $\mathbf{v}_{\mathrm{rel}} = \dot{x}\mathbf{a}_1$ and, because $\mathbf{a}_1 \cdot \mathbf{a}_2 \neq 0$, $\mathbf{F}_f$ appears in two of Lagrange's equations.
3. For Problem 2(d), the constraint force has three components and two equivalent representations:

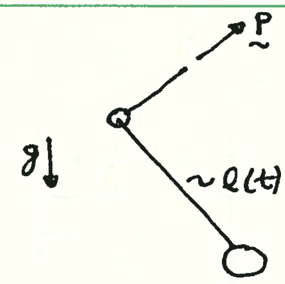
$$\mathbf{F}_c = \sum_{k=1}^{3} \lambda_k \mathbf{a}^k = \mathbf{N} + \mathbf{F}_f. \tag{4}$$

where

$$\mathbf{N} = \lambda_2\mathbf{a}^2 + \lambda_3\mathbf{a}^3, \qquad \mathbf{F}_f = F_f^1\mathbf{a}_1 \neq \lambda_1\mathbf{a}^1. \tag{5}$$

and $\|\mathbf{F}_f\| \leq \mu_s \|\mathbf{N}\|$.

QUESTION 1




(a)

$$\Psi = R - \ell = 0$$

$$\underline{F_c} = \lambda \nabla \Psi = -S\underline{e_R}$$

$S = -\lambda =$ tension in cable

$\nabla \Psi = \underline{e_R}$

(b) $$T = \tfrac{1}{2} m \left( \dot{R}^2 + R^2 \sin^2\phi\, \dot{\theta}^2 + R^2 \dot{\phi}^2 \right) \qquad U = mgR\cos\phi$$

$$\underline{F} = -mg\underline{E_3} - S\underline{e_R}$$

EOM

$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} = m\ell^2\dot{\phi} \right) - \left( \frac{\partial T}{\partial \phi} = mR^2 \sin\phi\cos\phi\, \dot{\theta}^2 = m\ell^2 \sin\phi\cos\phi\, \dot{\theta}^2 \right)$$

$$= \underline{F}\cdot \ell\, e_\phi = mg\ell\sin\phi$$

$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{\theta}} = mR^2\sin^2\phi\, \dot{\theta} = m\ell^2\sin^2\phi\, \dot{\theta} \right) - \left( \frac{\partial T}{\partial \theta} = 0 \right)$$

$$= \underline{F}\cdot \ell\sin\phi\, \underline{e_\theta} = 0$$

EOM are

$$m\ell^2\ddot{\phi} + 2m\ell\dot{\ell}\dot{\phi} - m\ell^2\sin\phi\cos\phi\,\dot{\theta}^2 = mg\ell\sin\phi$$

$$\frac{d}{dt}\left( m\ell^2 \sin^2\phi\, \dot{\theta} \right) = 0$$

(c) $$\dot{E} = \underline{F_{nc}}\cdot\underline{V} \quad \text{where} \quad E = T + U$$

$$= -Se_R\cdot\underline{V} = -S\dot{\ell} \neq 0 \quad \Rightarrow E \text{ is not conserved}$$

$$\underline{H_o}\cdot\underline{E_3} = m\ell^2\sin^2\phi\,\dot{\theta} \quad \Rightarrow$$ From second equation of motion $\underline{H_o}\cdot\underline{E_3}$ is conserved.

TOPS

(d)

$\|\underline{P}\| = |S|$ to determine $S$ we look at the $R$ equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}} = m\dot{R} = m\dot{\ell}\right) - \left(\frac{\partial T}{\partial R} = mR(\dot{\phi}^2 + \dot{\theta}^2 \sin^2\phi) = m\ell(\dot{\phi}^2 + \dot{\theta}^2 \sin^2\phi)\right)$$

$$= \underline{F}\cdot\underline{e}_R$$

$$= S - mg\cos\phi$$

Hence

$$\|\underline{P}\| = |S| = \left| m\ddot{\ell} + mg\cos\phi - m\ell(\dot{\phi}^2 + \dot{\theta}^2 \sin^2\phi)\right|$$

TOPS

QUESTION 2

$$q^1 = x$$
$$q^2 = y - A\cosh\left(\frac{x-x_0}{\ell}\right)$$
$$q^3 = z$$

$$x = q^1, \quad y = q^2 + A\cosh\left(\frac{q^1-x_0}{\ell}\right), \quad z = q^3$$

(a)

$$\underline{a}^1 = \nabla q^1 = \underline{E}_1$$
$$\underline{a}^2 = \nabla q^2 = \underline{E}_2 - \frac{A}{\ell}\sinh\left(\frac{x-x_0}{\ell}\right)\underline{E}_1$$
$$\underline{a}^3 = \nabla q^3 = \underline{E}_3$$

$$[a^{ik}] = \begin{bmatrix} 1 & -\frac{A}{\ell}\sinh\left(\frac{x-x_0}{\ell}\right) & 0 \\ -\frac{A}{\ell}\sinh\left(\frac{x-x_0}{\ell}\right) & 1+\frac{A^2}{\ell^2}\sinh^2\left(\frac{x-x_0}{\ell}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\underline{r} = q^1\underline{E}_1 + \left(q^2 + A\cosh\left(\frac{q^1-x_0}{\ell}\right)\right)\underline{E}_2 + q^3\underline{E}_3$

$$\underline{a}_1 = \frac{\partial \underline{r}}{\partial q^1} = \underline{E}_1 + \frac{A}{\ell}\sinh\left(\frac{q^1-x_0}{\ell}\right)\underline{E}_2$$
$$\underline{a}_2 = \underline{E}_2 = \frac{\partial \underline{r}}{\partial q^2}$$
$$\underline{a}_3 = \frac{\partial \underline{r}}{\partial q^3} = \underline{E}_3$$

(c) The constraints on the particle are $q^2 = y_0$ and $q^3 = 0$

The constraint force is $\underline{F}_c = \lambda_1 \underline{a}^2 + \lambda_2 \underline{a}^3 - \mu_K \|\underline{N}\| \dfrac{\dot{q}^1 \underline{a}_1}{\|\dot{q}^1 \underline{a}_1\|}$

Here $\underline{N}$ = normal force = $\lambda_1 \underline{a}^2 + \lambda_2 \underline{E}_3$

$\dot{q}^1 \underline{a}_1$ = velocity of particle on curve.

$$T = \tfrac{1}{2} m \underline{v} \cdot \underline{v}$$

where $\underline{v} = \dot{q}^1 \underline{a}_1 + \dot{q}^2 \underline{E}_2 + \dot{q}^3 \underline{E}_3$  $q^1 = x,\ q^2 = \eta,\ q^3 = z$

$$T = \tfrac{1}{2} m \left( \dot{x}^2 + \left( \dot{\eta} + \dot{x} \frac{A}{\ell} \sinh\left( \frac{x - x_0}{\ell} \right)\right)^2 + \dot{z}^2 \right)$$

EOM (need to use Approach I)

$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{x}} = m\dot{x}\left( 1 + \frac{A^2}{\ell^2} \sinh^2\left( \frac{x - x_0}{\ell} \right)\right)\right)$$

$$- \left( \frac{\partial T}{\partial x} = m\dot{x} \frac{A^2}{\ell^2} \sinh\left( \frac{x - x_0}{\ell} \right) \cosh\left( \frac{x - x_0}{\ell} \right)\right)$$

$$= \underline{F} \cdot \underline{a}_1 = -\frac{mgA}{\ell} \sinh\left( \frac{x - x_0}{\ell} \right) - \mu_K \|\underline{N}\| \frac{\dot{q}^1 a_{11}}{\|\dot{q}^1 \underline{a}_1\|}$$

where $a_{11} = \underline{a}_1 \cdot \underline{a}_1 = 1 + \dfrac{A^2}{\ell^2} \sinh^2\left( \dfrac{x - x_0}{\ell} \right)$

$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{\eta}} = m \dot{x} \frac{A}{\ell} \sinh\left( \frac{x - x_0}{\ell} \right)\right) - \left( \frac{\partial T}{\partial \eta} = 0 \right)$$

$$= \underline{F} \cdot \underline{a}_2 = -mg + \lambda_1$$

$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{z}} = m\dot{z} = 0 \right) - \left( \frac{\partial T}{\partial z} = 0 \right) = \underline{F} \cdot \underline{a}_3 = \lambda_2$$

Equation of motion

let $\frac{A}{\ell}\sinh\left(\frac{x-x_0}{\ell}\right) = f_x$ , $f_{xx} = \frac{A}{\ell^2}\cosh\left(\frac{x-x_0}{\ell}\right)$

$$\frac{d}{dt}\left(m\dot{x}(1+f_x f_x)\right) - m\dot{x} f_x f_{xx} = -mg f_x - \mu_K \|\lambda_1 \underline{a}^2\| \frac{\dot{x}a_{11}}{\|\dot{x}\underline{a}_1\|}$$

where $\lambda_1 = mg + \frac{d}{dt}\left(m\dot{x}\frac{A}{\ell}\sinh\left(\frac{x-x_0}{\ell}\right)\right)$

(d) Particle is stationary. then $\underline{F} = \underline{0}$

$$\lambda_1 \underline{a}^2 + \lambda_2 \underline{a}^3 + F_f \underline{a}_1 = mg\,\underline{E}_2$$

Solving for $\lambda_1, \lambda_2$ and $F_f$

$$\lambda_1 a^{22} + 0 + 0 = mg\,\underline{E}_2 \cdot \underline{a}^2 = mg \,, \quad a^{22} = 1 + f_x f_x$$

$$\lambda_2 + 0 + 0 = mg\,\underline{E}_2 \cdot \underline{a}_3 = 0$$

$$F_f a_{11} + 0 + 0 = mg\,\underline{E}_2 \cdot \underline{a}_1 = mg f_x \,, \quad a_{11} = 1 + f_x f_x$$

Hence

$$\underline{N} = \frac{mg}{1+f_x f_x}\underline{a}^2$$

$$\underline{F}_f = \frac{mg f_x}{1+f_x f_x}\underline{a}_1$$