EECS 126 Probability and Random Processes University of California, Berkeley: Spring 2019
Kannan Ramchandran February 19, 2019

Midterm 1

Last Name	First Name	SID

***Rules*.**

- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- You have 10 minutes to read the exam and 90 minutes to complete it.
- The exam is not open book; we are giving you a cheat sheet. No calculators or phones allowed.
- Unless otherwise stated, all your answers need to be justified. Show all your work to get partial credit.
- Maximum you can score is 114 but 100 points is considered perfect.

Problem	points earned	out of
Problem 1		54
Problem 2		10
Problem 3		25
Problem 4		25
Total		114

Problem 1: Answer these questions briefly but clearly.

(a) **[6]** Show that if A and B are events with $\mathbb{P}(A) = \mathbb{P}(B) = 1/2$, then $\mathbb{P}(A \cap B) = \mathbb{P}(A^c \cap B^c)$

We note that $\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B) = \mathbb{P}(A\cup B) = 1-\mathbb{P}(A^c \cap B^c)$. Thus, $\mathbb{P}(A\cap B) = \mathbb{P}(A^c\cap B^c)$ as desired.

(b) **[6]** Bob plays the following game: He tosses independent coins (which have probability of heads as p) until he gets a heads. Let N be the number of coins he tosses. He then re-tosses all those coins, and counts the number of heads among them, let that be X. What is $\mathbb{E}\,X$? Write your final answer in the box.

$\boxed{}$

We note that X conditioned on N is a binomial with parameter p, so it has expectation Np. Since N is geometric, N has expectation $\frac{1}{p}$. Now, we use double conditioning as $\mathbb{E}[\mathbb{E}[X \mid N]] = \mathbb{E}[Np] = 1$.

(c) **[6]** Find the distribution of Y (denoted $f_Y(y)$ for $y \in (-1,1)$) if X is uniform on $[-\frac{3\pi}{2}, \frac{3\pi}{2}]$ and $Y = \sin(X)$. Note that the inverse of the sine function ($\arcsin(x) : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$) has a derivative of $\frac{1}{\sqrt{1-x^2}}$.

Start by noting that X being uniform on $[-\frac{3\pi}{2}, \frac{3\pi}{2}]$ and $[-\pi/2, \pi/2]$ are exactly the same by symmetry for this problem. (Alternatively, use the law of total probability, and condition on X being in the interval $[-\pi/2, \pi/2]$, and then condition it on $[\pi/2, 3\pi/2]$, and then condition on $[-\pi/2, -3\pi/2]$. Drawing the graph of $\sin(x)$ in this interval, we notice that the cdf of Y conditioned on each of these intervals is the same.)

For $y \in (-1,1)$, $\mathbb{P}(Y \le y) = \mathbb{P}(X \le \sin^{-1}(y)) = (\sin^{-1}(y))/\pi$, so $f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}$.

(d) **[6]** Find a pair of random variables X, Y such that X, Y are dependent, but X^2 and Y^2 are independent.

Let U be uniform on $\{-1, 1\}$ (also referred to as a Rademacher random variable), and let V, W be independent RVs that are not a.s. 0. (Say, Bernoulli(1/2)) Let $X = UV$, and $Y = UW$. Now, these are clearly dependent ($\mathbb{P}(X = -1 | Y = 1) = 0 \neq \mathbb{P}(X = -1)$), but when squared U disappears.

(e) **[6]** There are N ($\geq$ 2) servers behind a counter at a store, having independent service times that are exponentially distributed with rates $\lambda_1, \lambda_2, \ldots, \lambda_N$ respectively. There is a single line (queue) for service to all N servers. Alice and Bob are customers who come into the store when all servers are busy serving their respective customers. Alice is first in line to be served next, followed by Bob. What is the probability that Alice and Bob get served by different servers? What are the expected departure times for Alice and Bob?

To calculate the probability that Alice and Bob get served by different servers, we can calculate the complement, the probability they get served by the same server. To do this, we condition on which server Alice gets. Let A, B take values in the set $S_1 \ldots S_N$ denoting the event that they get the corresponding server. Note, by the memoryless property of Poisson interarrival times, the probability Bob gets the ith server is still $\frac{\lambda_i}{\sum_{j=1}^N \lambda_j}$.

$$\begin{aligned}
P(\text{different servers}) &= 1 - P(\text{same servers}) \\
&= 1 - \sum_{i=1}^{N} P(B = S_i | A = S_i) P(A = S_i) \\
&= 1 - \sum_{i=1}^{N} \frac{\lambda_i}{\sum_{j=1}^N \lambda_j} * \frac{\lambda_i}{\sum_{j=1}^N \lambda_j} \\
&= 1 - \sum_{i=1}^{N} (\frac{\lambda_i}{\sum_{j=1}^N \lambda_j})^2 \\
&= 1 - \frac{\sum_{i=1}^N (\lambda_i^2)}{(\sum_{j=1}^N \lambda_j)^2}
\end{aligned}$$

For the expected departure times, we can use the memoryless property between the wait time and time for Alice to get served. We know that Alice's departure time is equal one arrival time plus the expected time she takes to get served. Similarly, Bob's departure time is equal to two arrival times plus his serving time. To compute the expected time for Alice and Bob to get served, we use iterated expectation on which server gets to Alice. Let T_A, T_B denote the departure times for Alice and Bob respectively.

$$\begin{aligned}
\mathbb{E}[T_A] &= \mathbb{E}[\text{time until Alice gets served}] + \mathbb{E}[\text{time for Alice's service}] \\
&= \frac{1}{\sum_{i=1}^N \lambda_i} + \sum_{i=1}^{N} \mathbb{E}[\text{time for Alice's service} | A = S_i] \mathbb{P}(A = S_i) \\
&= \frac{1}{\sum_{i=1}^N \lambda_i} + \sum_{i=1}^{N} \frac{1}{\lambda_i} \frac{\lambda_i}{\sum_{j=1}^N \lambda_j} \\
&= \frac{N+1}{\sum_{i=1}^N \lambda_i}
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[T_B] &= \mathbb{E}[\text{time until Alice gets served}] + \mathbb{E}[\text{time until Bob gets served}] + \mathbb{E}[\text{time for Bob's service}] \\
&= \frac{1}{\sum_{i=1}^{N}\lambda_i} + \frac{N+1}{\sum_{i=1}^{N}\lambda_i} \\
&= \frac{N+2}{\sum_{i=1}^{N}\lambda_i}
\end{aligned}
$$

(f) [6] I have 10 independent draws from a U(0,1) distribution (uniform distribution between 0 and 1). What is the expected difference between the largest and the smallest draws? What is the probability that the third largest draw is smaller than 0.8 given that the largest draw is 0.9?

Let X_{min} be the value of the smallest draw and X_{max} be the value of the largest draw. By linearity of expectation, $\mathbb{E}[X_{max} - X_{min}] = \mathbb{E}[X_{max}] - \mathbb{E}[X_{min}]$. The CDF for X_{max} is given by

$$\mathbb{P}(X_{max} \le x) = \mathbb{P}(X_1, X_2, ..., X_{10} \le x) = \mathbb{P}(X_1 \le x) \cdot ... \cdot \mathbb{P}(X_{10} \le x) = \mathbb{P}(X_1 \le x)^{10} = x^{10}$$

, where the second equality used independence and the third equality used the fact that they are i.i.d. Using the tail sum formula, we have that

$$\mathbb{E}[X_{max}] = \int_0^1 \mathbb{P}(X_{max} > x)dx = \int_0^1 (1 - x^{10})dx = 1 - \frac{x^{11}}{11}\Big|_0^1 = \frac{10}{11}$$

By a similar calculation we can find that $\mathbb{E}[X_{min}] = \frac{1}{11}$, so the answer is $\frac{9}{11}$.

For the second part, we note that since we have drawn from a uniform distribution, if we condition on the fact that the largest draw is 0.9, we are now interested in the probability that the second highest draw of nine draws from a U(0,0.9) distribution is less than 0.8. This can be split into two disjoint events: one where all 9 draws are less than 0.8 and another where the highest draw is greater than 0.8. Thus, the probability is $\binom{9}{1}(\frac{0.8}{0.9})^8(\frac{0.1}{0.9}) + (\frac{0.8}{0.9})^9$

(g) [6] You have N items, G of which are good and B of which are bad. You start to draw items without replacement, and suppose that the first good item appears on draw X. Find $\mathbb{P}(X > k)$.

Number the bad items $B_1, ..., B_b$ and the good items $G_1, ..., G_g$. The event that $X > k$ is exactly the event that the first k items picked are all bad. Let $A_k(B_{i_1}, ..., B_{i_k})$ be the event that first k balls picked are $B_{i_1}, ..., B_{i_k}$.

$$\mathbb{P}(X > k) = \sum_{(i_1, i_2, ..., i_k)} \mathbb{P}(A_k(B_{i_1}, ..., B_{i_k}))$$

By symmetry, each term in the above summation has the same value. So we have:

$$\mathbb{P}(X > k) = \binom{B}{k} k! \cdot \mathbb{P}(A_k(B_1, ..., B_k))$$

Now, compute the probability that the first k items picked are $B_1, ..., B_k$. Fix these items in the first k positions. Then there are $(N-k)!$ ways of permuting the rest of the items. All $N!$ permutations are equally likely, so this probability is $(N-k)!/N!$.

$$\mathbb{P}(X > k) = \binom{B}{k} \frac{k!(N-k)!}{N!} = \frac{\binom{B}{k}}{\binom{N}{k}}$$

Note: It is also possible to simply observe how to put items onto the first k slots: there are N choose k ways to put items and B choose k ways to put only bad items. This is a valid argument so long as you understand why all the possible outcomes for the first k draws are uniform.

(h) [6] Show that for a random variable that is upper bounded by B, the following probability bound holds:

$$P(X \leq t) \leq \frac{B - \mathbb{E}[X]}{B - t} \quad \text{for all} \quad t \leq B$$

$$P(X \leq t) = P(-X \geq -t) = P(B - X \geq B - t) \leq \frac{B - \mathbb{E}[X]}{B - t} \quad \text{for all} \quad t \leq B$$

We can use Markov's Inequality because the random variable $B - X$ is non-negative.

(i) [6] Consider the r.v.s $Y \sim \mathcal{N}(0,1)$ and $Z \sim Pois(\lambda)$.

Calculate $\mathbb{E}[Y^Z]$

Solution 1: Iterated expectation conditioning on Y:

$$\mathbb{E}[Y^Z | Y = y] = \mathbb{E}[y^Z] = \sum_{z=0}^{\infty} y^z \cdot \mathbb{P}(Z = z) = \sum_{z=0}^{\infty} y^z \cdot \frac{\lambda^z e^{-\lambda}}{z!}$$

$$= e^{-\lambda} \cdot \sum_{z=0}^{\infty} \frac{(\lambda y)^z}{z!} = e^{-\lambda} \cdot e^{\lambda y} = e^{\lambda(y-1)} \implies \mathbb{E}[X|Y] = e^{\lambda(Y-1)}$$

Using iterated expectation,

$$\mathbb{E}[Y^Z] = \mathbb{E}[\mathbb{E}[Y^Z|Y]] = \mathbb{E}[e^{\lambda(Y-1)}] = M_{Y-1}(\lambda)$$

But since $Y \sim \mathcal{N}(0,1)$, $Y - 1 \sim \mathcal{N}(-1,1)$, so the expectation is the MGF of this which is $e^{\frac{\lambda^2}{2} - \lambda}$

Solution 2: Iterated expectation conditioning on Z:
Recalling Gaussian moments from hw3:

$$\mathbb{E}[Y^Z | Z = z] = \mathbb{E}[Y^z] = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (z-1)!! = \frac{z!}{2^{\frac{z}{2}} (\frac{z}{2})!} & \text{if } n \text{ is even} \end{cases}$$

So

$$\mathbb{E}[Y^Z] = \mathbb{E}[\mathbb{E}[Y^Z|Z]] = \sum_{z=0}^{\infty} \frac{\cancel{z!}}{2^{\frac{z}{2}}(\frac{z}{2})!} \cdot \mathbb{1}(Z \text{ is even}) \cdot \frac{\lambda^z e^{-\lambda}}{\cancel{z!}}$$

Changing index with $j = z/2$,

$$e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{2^j j!} = e^{-\lambda} \cdot \sum_{j=0}^{\infty} \frac{(\lambda^2/2)^j}{j!} = e^{\frac{\lambda^2}{2} - \lambda}$$

Alternatively, even without the moments result from hw, we simply have

$$\mathbb{E}[\mathbb{E}[Y^Z|Z]] = \sum_{z=0}^{\infty} \mathbb{E}[Y^z] \frac{\lambda^z e^{-\lambda}}{z!} = e^{-\lambda} \underbrace{\sum_{z=0}^{\infty} \frac{\mathbb{E}[(\lambda Y)^z]}{z!}}_{M_Y(\lambda)} = e^{-\lambda} M_Y(\lambda)$$

Problem 2: Basic probability

(a) **[2]** Say you have n balls and n buckets numbered 1 to n. Now, you are going to put one ball in each bucket by picking a permutation uniformly at random. Let X be the number of balls that have the same number as their bin. What is $\mathbb{E}\,X$? Write your final answer in the box.

It is 1 by linearity of expectation.

(b) **[3]** Same set-up, but now instead you toss the balls uniformly at random at each bucket (now buckets can have more than one ball). Let Y be the number of balls that have the same number as their bin. What is $\mathbb{P}(Y = 0)$? Write your final answer in the box.

It is $(\frac{n-1}{n})^n$

(c) **[5]** Compare the variances of X and Y. Which one is bigger? Justify briefly but rigorously. (You do not need to compute the variances of X and Y to answer this problem.)

The first one has a much higher variance; that is because the variables are positively correlated. Let $\mathbb{E}[X_i X_j]$ be the indicators defined for the first problem. We then have $\mathbb{E}[X_i X_j] = 1$ if both i and j are fixed, which happens with probability $\frac{1}{n(n-1)} > \frac{1}{n^2}$.

Problem 3: Apples!

You are starting a business of shipping apples. On each day, you get X requests, where X is a Poisson random variable with a rate of 6 requests/day (so $\lambda = 6$).

(a) **[3]** For each request, you make a box and put Y_i apples in it where Y_i is uniform in the set $\{1, 2, ..., 6\}$ (and i goes from 1 to X) and all Y_is are independent of each other. What is the distribution of the number of boxes that you ship that have 6 apples in it?

It's Poisson with parameter $\frac{\lambda}{6} = 1$ by Poisson splitting.

(b) **[5]** When your customers start complaining that they got a different number of apples than their neighbor, you decide to change your business model a little bit. Every day, you decide to toss a fair 6 sided die, and put that many apples inside each of the shipments. (So, if you have 10 requests, and you roll a 4, you are going to give away 40 apples total.)

Now you start to get scared: It is possible that you give away way too many apples because on some day, you roll big and you have a lot of requests. More formally, let Z be the number of apples you give away. Using a Markov bound, upper bound the probability that Z is greater than or equal to 70.

Let Y be the number of apples in each box, so we have $Z = XY$. Since X and Y are independent, $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = 6\frac{7}{2} = 21$, so by Markov, we have $\mathbb{P}(XY \geq 70) \leq \frac{21}{70} = \frac{3}{10}$.

(c) **[5]** Using the Chebyshev bound, find the probability that $Z \geq 70$? (For simplicity, you can use the fact that the variance of a die roll comes down to $\frac{35}{12}$)

We use the definition of conditional variance for $\text{Var}(Z) = \text{Var}(XY) = \text{Var}(\mathbb{E}[XY|Y]) + \mathbb{E}[\text{Var}(XY|Y)]$. Now, knowing Y, XY is simply a scaled Poisson distribution, and we know how the expectation and variance scales, so $\text{Var}(XY) = \text{Var}(\mathbb{E}[XY|Y]) + \mathbb{E}[\text{Var}(XY|Y)] = \text{Var}(6Y) + \mathbb{E}[6Y^2]$. Now, we can find the second moment of Y as

$36\frac{35}{12} + 6\frac{91}{6} = 196$

Now, plugging it into the Chebyshev bound, we get that $\mathbb{P}(Z \geq 70) = \mathbb{P}(Z - \mathbb{E}\,Z \geq 49) = \frac{196}{49^2} = \frac{4}{49}$

(d) **[5]** Now, say that you are a customer of this business instead. You receive orders from this business every day; how many days will it take in expectation for you to receive all 6 different types of boxes? Leave it as a sum of fractions.

This is just a coupon-collector problem with 6 coupons, so the expected time is $6\sum_{i=1}^{6}\frac{1}{i}$.

(e) **[7]** Say that you are trying to deduce how many apples were shipped in your box based on how long the shipment takes. In particular, say that the arrival time of a box containing k apples is exponential with parameter k per hour (So, on average, it takes $\frac{1}{k}$ hours for a box of size k apples to arrive). What is the probability that your box contains 5 apples given that you have already waited 5 hours?

This is a simple Bayes problem; we have that $\mathbb{P}(X > 5|Z = 5)\mathbb{P}(Z = 5) = \mathbb{P}(Z = 5|X > 5)\mathbb{P}(X > 5)$

Now, to find $\mathbb{P}(X > 5)$, we divide it up to cases:

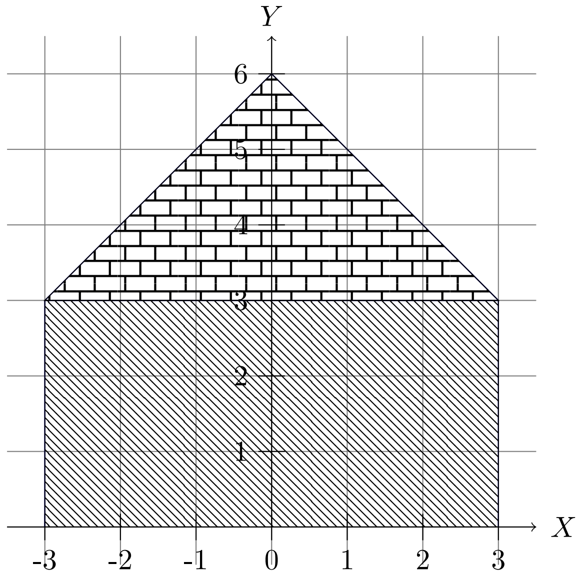
$$\mathbb{P}(X > 5) = \sum_k \mathbb{P}(X > 5|Z = k)\mathbb{P}(Z = k) = \frac{1}{6}\sum_k \mathbb{P}(X > 5|Z = k)\mathbb{P}(Z = k) = \frac{1}{6}\sum_k e^{-5k}$$

. We know $\mathbb{P}(X > 5|Z = 5) = e^{-5.5}$ and $\mathbb{P}(Z = 5) = 1/6$. So our final answer is -

$$\frac{e^{-25}}{\sum_{k=1}^{6} e^{-5k}}$$

Problem 4: Chimney construction

Shown below is a density profile of a house from which you are going to sample a point (X, Y). *Note that the density $f_{(X,Y)}$ of the brick roof ($\mathcal{B}$) is twice that of the wall of the house ($\mathcal{W}$) (check the shading).*



(a) **[5]** What is the joint PDF of (X, Y)?

Let D be the density inside the wall. Then,

$$\int_W D \, \mathrm{d}A + \int_B 2D \, \mathrm{d}A = 1$$

i.e. $D \cdot (\text{Area of } W) + 2D \cdot (\text{Area of } B) = 1$, so $D(18) + 2D(9) = 1$, and $D = \frac{1}{36}$. Therefore, the joint pdf is

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{36} & (x, y) \in \mathcal{W} \\ \frac{1}{18} & (x, y) \in \mathcal{B} \\ 0 & \text{else} \end{cases}$$

(b) **[10]** Plot the marginal PDFs of $f(x)$ and $f(y)$.

For $-3 \leq x \leq 3$,

$$f_X(x) = \int_0^{6-|x|} f_{X,Y}(x,y)\ \mathrm{d}y = \int_0^3 \frac{1}{36}\ \mathrm{d}y + \int_3^{6-|x|} \frac{1}{18}\ \mathrm{d}y = \frac{1}{12} + \frac{1}{18}(3-|x|)$$

So

$$f_X(x) = \begin{cases} \frac{1}{12} + \frac{1}{18}(3-|x|) & -3 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

For $0 \leq y \leq 3$,

$$f_Y(y) = \int_{-3}^{3} f_{X,Y}(x,y)\ \mathrm{d}x = \int_{-3}^{3} \frac{1}{36}\ \mathrm{d}x = \frac{1}{6}$$

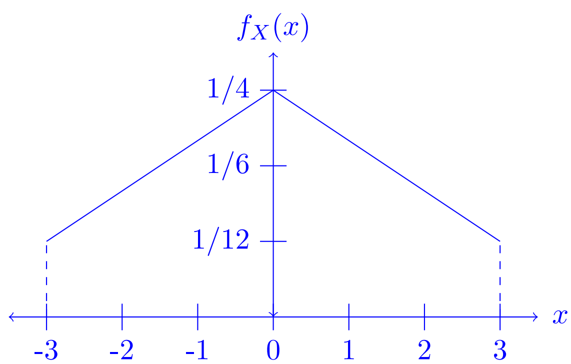
For $3 \leq y \leq 6$,

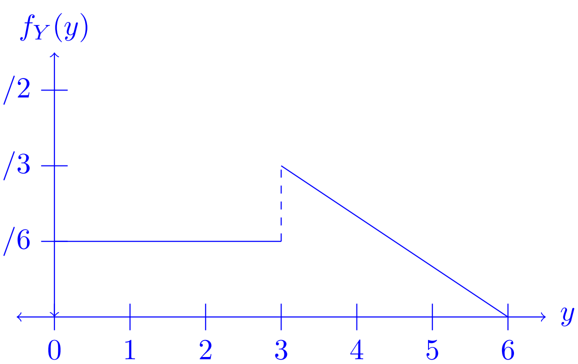
$$f_Y(y) = \int_{y-6}^{6-y} f_{X,Y}(x,y)\ \mathrm{d}x = \int_{y-6}^{6-y} \frac{1}{18}\ \mathrm{d}x = \frac{1}{9}(6-y)$$

So

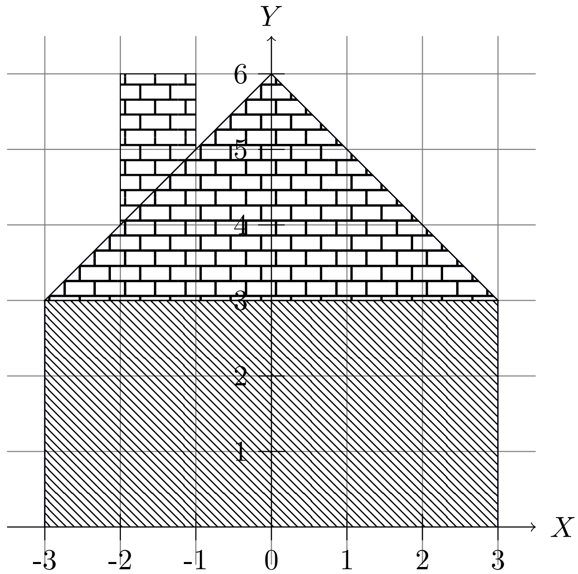
$$f_Y(y) = \begin{cases} \frac{1}{6} & 0 \leq y \leq 3 \\ \frac{1}{9}(6-y) & 3 < y \leq 6 \\ 0 & \text{else} \end{cases}$$

The plots for the marginal pdfs $f_X(x)$ and $f_Y(y)$ are shown below.





Now, you want to install a little chimney to your house as follows:



(c) **[10]** Find the expectation of X for the house with chimney.

First, we note that much like part (a) we have that $D = \frac{1}{39}$.

Let C be the event that the point chosen is in the chimney. $\mathbb{E}[X] = \mathbb{E}[X|C]\mathbb{P}(C) + \mathbb{E}[X|C^{\mathsf{c}}]\mathbb{P}(C^{\mathsf{c}})$.

$\mathbb{E}[X|C^{\mathsf{c}}] = 0$ by symmetry.

To find the expectation for the chimney, you can further divide it into a triangle (denote by Δ) and the square (denote by $\square$), which have conditional means of $-\frac{5}{3}$, and $-\frac{3}{2}$ respectively.

So, we get

$\mathbb{E}[X] = \mathbb{E}[X|C]\mathbb{P}(C) + \mathbb{E}[X|C^{\mathsf{c}}]\mathbb{P}(C^{\mathsf{c}}) = (\mathbb{E}[X|\Delta]P(\Delta) + \mathbb{E}[X|\square]P(\square)) + 0 = \frac{-5}{3}\frac{1}{39} + \frac{3}{2}\frac{2}{39} = \frac{-14}{117}$