ME 132, Fall 2018, Quiz # 2

# 1	# 2	# 3	# 4	#5	NAME
20	15	30	20	15	

Rules:

1. Two (2) sheet of notes allowed. 8.5 × 11 inches. Both sides can be used.
2. Calculator is allowed.
3. **No laptops, phones, headphones, pads, tablets, or any other such device may be out. If such a device is seen after 10:10AM, your test will be confiscated, and you will get a 0 for the exam.**
4. **Keep your eyes on your own paper!**
5. **The exam ends promptly at 11:00 AM.**
6. **Stop working, and turn in exams when notified.**

For complex numbers N and D

$$|ND| = |N| \cdot |D|, \quad \angle(ND) = \angle N + \angle D$$

Furthermore, if $D \neq 0$,

$$\angle \frac{1}{D} = -\angle D.$$

If D is real, and $D > 0$, then $\angle D = 0$.

1. Consider the 2×2 matrix, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(a) Show that

$$v_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are eigenvectors of A, **and determine the corresponding eigenvalues**, denoted as λ_1 and λ_2.

$$Av_1 = \lambda_1 v_1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \underline{\lambda_1 = 1}$$

$$Av_2 = \lambda_2 v_2 \rightarrow \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \underline{\lambda_2 = -1}$$

(b) Find an invertible matrix $V \in \mathbb{R}^{2\times 2}$ and diagonal matrix $\Lambda \in \mathbb{R}^{2\times 2}$ such that $AV = V\Lambda$.

V is matrix formed by eigenvectors (as columns)
Λ is diagonal, with eigenvalues on diagonal

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad V = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \ \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) Find the expression for e^{At}.

$$A = V\Lambda V^{-1} \Rightarrow e^{At} = V e^{\Lambda t} V^{-1}$$

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow V^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

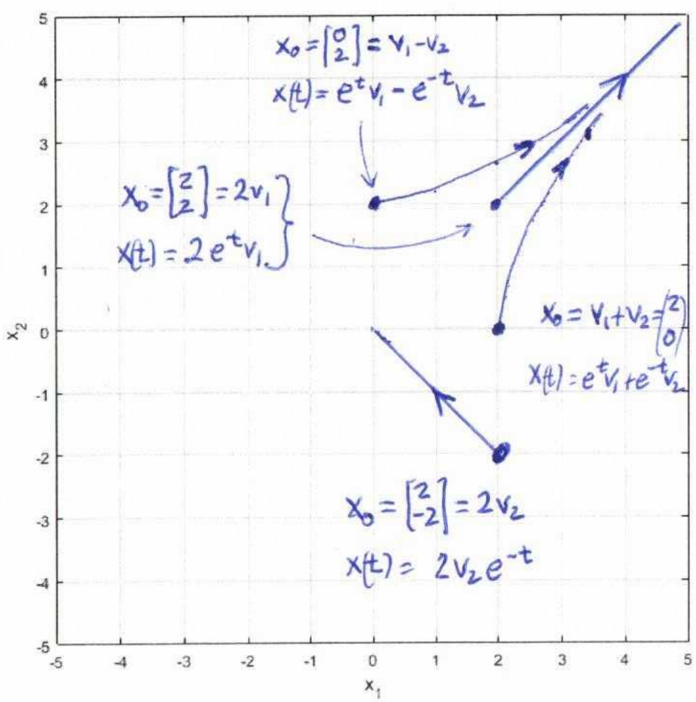
$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{t} & \\ & e^{-t} \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}$$

(d) Consider the differential equation $\dot{x}(t) = Ax(t)$ with initial conditions

$$x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

In the 2-dimensional "phase-plane" below, sketch the solutions for $t \geq 0$, starting from these 4 different initial conditions. Include arrows to show the direction as t increases



2. For the linear system $\dot{x}(t) = Ax(t) + Bu(t)$, the matrices A and B are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Consider a state-feedback, $u(t) = Kx(t)$, where $K = [K_1\ K_2\ K_3] \in \mathbb{R}^{1\times 3}$. Find A_c such that the closed-loop dynamics are $\dot{x}(t) = A_c x(t)$. Note that A_c should depend on numerical values in A and B, as well as the entries that make up K.

$$\dot{x}(t) = Ax(t) + B\underbrace{u(t)}_{Kx(t)} = Ax(t) + BKx(t) = \underbrace{(A+BK)}_{A_c}x(t)$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} = \begin{bmatrix} K_1 & 1+K_2 & K_3 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

(b) Find the closed-loop characteristic polynomial?

$$\det(\lambda I_3 - A_c) = \det\begin{bmatrix} \lambda - K_1 & -(1+K_2) & -K_3 \\ -1 & \lambda & 0 \\ 1 & 1 & \lambda \end{bmatrix}$$

$$= (\lambda - K_1)\lambda^2 + (-K_3)(-1)(1) - [-(1+K_2)(-1)\lambda] - (-K_3)\lambda$$

$$= \lambda^3 - K_1\lambda^2 + (K_3 - K_2 - 1)\lambda + K_3$$

(c) For values for the entries of K such that the closed-loop eigenvalues are at $\{-1+j,\ -1-j,\ -2\}$

desired poly $= (\lambda+1-j)(\lambda+1+j)(\lambda+2)$

$= (\lambda^2 + 2\lambda + 2)(\lambda+2)$

$= \lambda^3 + 4\lambda^2 + 6\lambda + 4$

match coefficients: $-K_1 = 4, \quad K_3 - K_2 - 1 = 6, \quad K_3 = 4$

$$\boxed{K_1 = -4, \quad K_2 = -3, \quad K_3 = 4}$$

3. Consider the process model $\dot{x}(t) = x(t) + u(t) + d(t)$, $y(t) = x(t)$, where u is the control input, d is the disturbance, and y is the measured output. For simplicity, **ignore sensor noise in this problem.**

(a) Using the controller architecture

$$\begin{aligned} \dot{z}(t) &= r(t) - y(t) \\ u(t) &= K_I z(t) + K_0 r(t) + K_1 y(t) \end{aligned}$$

find the closed-loop state space matrices $\bar{A} \in \mathbb{R}^{2\times 2}, \bar{B} \in \mathbb{R}^{2\times 2}, \bar{C} \in \mathbb{R}^{1\times 2}, \bar{D} \in \mathbb{R}^{1\times 2}$ such that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \bar{A} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \bar{B} \begin{bmatrix} r(t) \\ d(t) \end{bmatrix}$$

and

$$y(t) = \bar{C} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \bar{D} \begin{bmatrix} r(t) \\ d(t) \end{bmatrix}.$$

substitute for u in $\dot{x}$ eq, and $y=x$ in u & $\dot{z}$ eqs, giving

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 1+K_1 & K_I \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} K_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ d(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ d(t) \end{bmatrix}$$

(b) In terms of K_I, K_0, K_1, what is the closed-loop characteristic polynomial?

$$\lambda_{clp} = \det\left(\lambda I_2 - \begin{bmatrix} (1+K_1) & K_I \\ -1 & 0 \end{bmatrix}\right) = \det \begin{bmatrix} \lambda-1-K_1 & -K_I \\ 1 & \lambda \end{bmatrix}$$

$$= \lambda^2 - (1+K_1)\lambda + K_I$$

(c) Design K_I and K_1 so that the closed-loop eigenvalues are a complex-conjugate pair, with ($\xi = 0.7, \omega_n = 2$).

$$\lambda_{des} = \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = \lambda^2 + 2.8\lambda + 4$$

match coeffs $\quad -(1+K_1) = 2.8 \quad K_I = 4 \quad \boxed{K_1 = -3.8 \quad K_I = 4}$

(d) Choose K_0 so that the instantaneous-gain from $r \to u$ is 0.5.

$$u(t) = K_I z(t) + K_0 r(t) + K_1 x(t)$$

$$IG_{r\to u} = K_0 \qquad \text{set } \boxed{K_0 = 0.5}$$

(e) For this closed-loop system, what is the steady-state gain from $r \to y$?

PI architecture for 1^{st} order plant $\Rightarrow$ $SSG_{r\to y} = 1$

(f) For this closed-loop system, what is the steady-state gain from $d \to y$?

PI architecture for 1^{st} order plant $\Rightarrow$ $SSG_{d\to y} = 0$

(g) With the K_I, K_0, K_1 already designed (and fixed at these values), suppose the process model changes to $\dot{x}(t) = 1.1x(t) + 0.9u(t) + 0.8d(t)$. **For the closed-loop system with this modified process.**

robust to changes

i. what is the steady-state gain from $r \to y$?

$(SSG_{r\to y} = 1)$ is robust to changes in plant <u>if</u> closed-loop remains stable. See part(iii)

ii. what is the steady-state gain from $d \to y$?

$(SSG_{d\to y} = 0)$ is robust to changes in plant <u>if</u> closed-loop remains stable. See part(iii)

closed-loop eigenvalues <u>do</u> change

iii. what are the closed-loop eigenvalues? with modifications, we have

$$A_{clp} = \begin{bmatrix} 1.1-(.9)(3.8) & (.9)4 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2.32 & 3.6 \\ -1 & 0 \end{bmatrix}$$

$$\text{char poly} = \lambda^2 + 2.32\lambda + 3.6$$

evals $\{-1.16 \pm j\,1.502\}$ $(\xi = .611, \omega_n = 1.9)$

$\rightarrow$ modified from $\xi = .7, \omega_n = 2$

4. This problem focuses on the response of the system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$. For each system given below

- Compute the eigenvalues of A
- Determine $\lim_{t\to\infty} y(t)$, for the response with $x(0) = 0_2$, $u(t) \equiv 1$ for all $t \geq 0$
- Determine $\dot{y}(0)$, for the response with $x(0) = 0_2$, $u(t) \equiv 1$ for all $t \geq 0$
- Make an approximate sketch of the response $y(t)$ versus t, with $u(t) \equiv 1$ for all $t \geq 0$, starting from $x(0) = 0_2$. **Numerically mark/label the vertical and horizontal scales.**

(a) $A = \begin{bmatrix} -5 & 4 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 3 \end{bmatrix}$

- Eigenvalues =

$\det(\lambda I - A) = (\lambda+5)(\lambda-1) + 8 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$

evals $\{-1, -3\}$

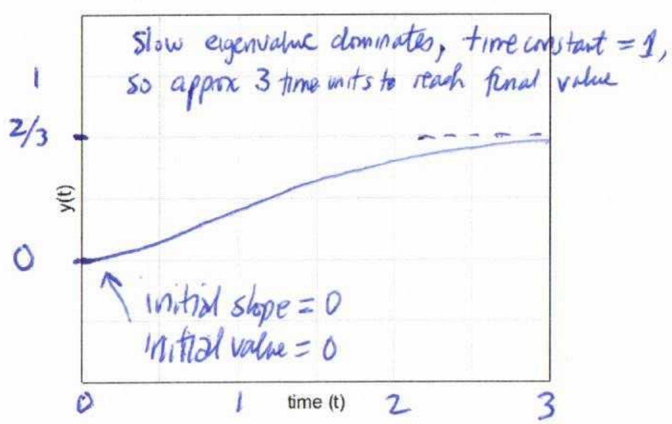
- $\lim_{t\to\infty} y(t) = -CA^{-1}B$

$-\begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} -5 & 4 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & -4 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2/3$

- $\dot{y}(0) = CB$

$= \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0$

- Sketch of response (mark/label both scales)



(b) $A = \begin{bmatrix} -1 & 5 \\ -5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 0 \end{bmatrix}$

- Eigenvalues =

$-1 \pm 5j$ time to decay ≈ 3 Visible periods ≈ 2.5

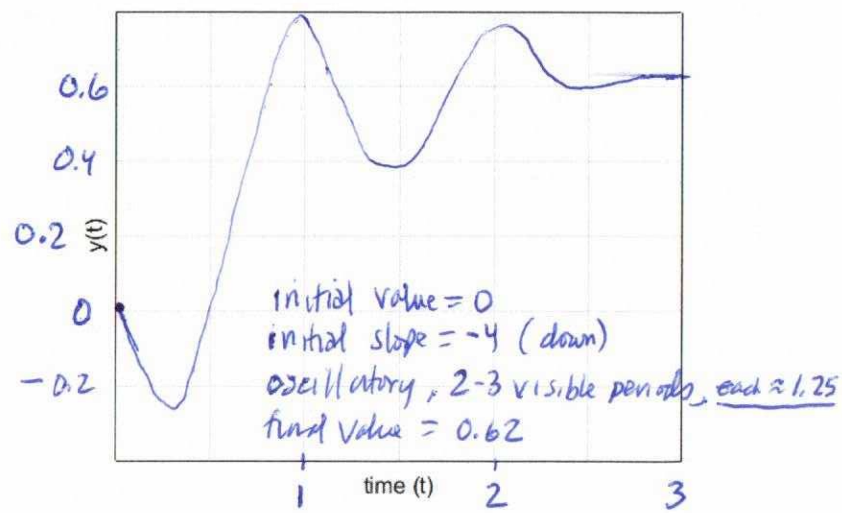
period of osc $= \frac{2\pi}{5}$

- $\lim_{t\to\infty} y(t) = -CA^{-1}B$

$\begin{bmatrix} -4 & 0 \end{bmatrix} \frac{1}{26} \begin{bmatrix} -1 & -5 \\ 5 & -1 \end{bmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{26} \begin{bmatrix} 4 & 20 \end{bmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{16}{26} = \frac{8}{13} \approx 0.62$

- $\dot{y}(0) = CB = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -4$

- Sketch of response (mark/label both scales)



5. Consider the 2-state dynamic system $\dot{x}(t) = Ax(t) + Bu(t)$, with

$$A := \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}, \quad B := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) What are the eigenvalues of A

$$\det(\lambda I - A) = (\lambda-4)(\lambda+4)+15 = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$$

$$\text{evals} = \{-1, 1\}$$

(b) Is the system stable? no

(c) Consider a state-feedback $u(t) = Kx(t)$ for $K \in \mathbb{R}^{1\times 2}$. Using this feedback, find the closed-loop differential equations governing x.

$$A+BK = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} 4+K_1 & -5+K_2 \\ 3+K_1 & -4+K_2 \end{bmatrix} x(t)$$

(d) What is the closed-loop characteristic equation?

$$\lambda_{clp} = \det\left(\lambda I_2 - \begin{bmatrix} 4+K_1 & -5+K_2 \\ 3+K_1 & -4+K_2 \end{bmatrix}\right)$$

$$= (\lambda-4-K_1)(\lambda+4-K_2) - (5-K_2)(-3-K_1)$$

$$= \lambda^2 + 4\lambda - K_2\lambda - 4\lambda - 16 + 4K_2 - K_1\lambda - 4K_1 + K_1K_2 + 15 + 5K_1 - 3K_2 - K_1K_2$$

$$= \lambda^2 + (4-K_2-4-K_1)\lambda + (-16+4K_2-4K_1+15+5K_1-3K_2)$$

$$= \lambda^2 + (-K_1-K_2)\lambda + (-1+K_1+K_2)$$

(e) Can you choose K so that the closed-loop eigenvalues are at $\{-4, -5\}$? Explain your answer.

$$\lambda_{des} = (\lambda+4)(\lambda+5) = \lambda^2 + 9\lambda + 20$$

match coeffs: $\left.\begin{aligned} -K_1-K_2 &= 9 \\ -1+K_1+K_2 &= 20 \end{aligned}\right)$ no solution, as dependence on K_1, K_2 is the same (not linearly independent) and no solution exists.