Problem 1 (50 points) Consider a DT-LTI filter F whose impulse response is given by

$$\forall n \in \mathbb{Z}, \quad f(n) = \delta(n) + \delta(n-1) + \delta(n-2)$$

(a) (20 points) The frequency response of the filter can be expressed as follows:

$$\forall \omega \in \mathbb{R}, \quad F(\omega) = A(\omega) e^{i\alpha\omega}$$

where $A(\omega) = 1 + 2\cos(\omega)$ and α is a real-valued quantity.

(i) (5 points) Determine α explicitly.

A plot of $f(n)$ will show its region of support is $\mathcal{S} \triangleq \{0, 1, 2\}$ and $f(n) = 1$ for $n \in \mathcal{S}$.

$$\begin{aligned} F(\omega) &= \sum_{n=-\infty}^{\infty} f(n)\, e^{-i\omega n} && (1) \\ &= e^{-i\omega(0)} + e^{-i\omega} + e^{-i2\omega} && (2) \end{aligned}$$

Expanding $F(\omega) = A(\omega)\, e^{i\alpha\omega}$ into complex exponentials, we get

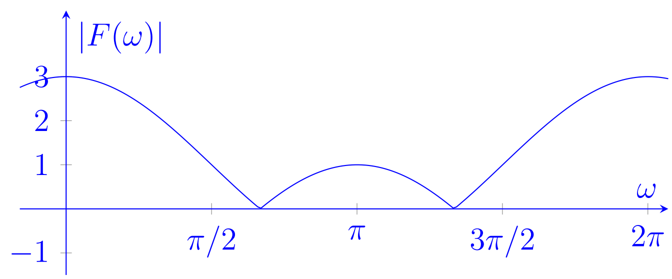
$$\begin{aligned} F(\omega) &= A(\omega)\, e^{i\alpha\omega} && (3) \\ &= (1 + 2\cos(\omega))\, e^{i\alpha\omega} && (4) \\ &= \left(1 + 2\frac{e^{i\omega} + e^{-i\omega}}{2}\right) e^{i\alpha\omega} && (5) \\ &= e^{i\alpha\omega} + e^{i(\alpha+1)\omega} + e^{i(\alpha-1)\omega} && (6) \end{aligned}$$

Now, we can pattern-match: the exponential with the smallest factor multiplying $i\omega$ in (2) is $e^{i\omega(-2)}$, and the corresponding exponential in (6) is $e^{i\omega(\alpha-1)}$, so $-2 = \alpha - 1 \implies \alpha = -1$. You can plug $\alpha = -1$ back into (6) to verify the two expressions for $F(\omega)$ match.

(b) (10 points) Provide a well-labeled plot of the magnitude response $|F(\omega)|$.

$$\begin{aligned} |F(\omega)| &= \left|(1 + 2\cos(\omega))\, e^{-i\omega}\right| && (7) \\ &= |1 + 2\cos(\omega)| \underbrace{\left|e^{-i\omega}\right|}_{1} && (8) \\ &= |1 + 2\cos(\omega)| && (9) \end{aligned}$$

Since $A(\omega)$ is real-valued, the modulus operator $|\cdot|$ in (9) is just an absolute value.



(c) (5 points) Let the input to the filter be

$$\forall n \in \mathbb{Z}, \quad x(n) = 1 + (-1)^n + \cos\left(\frac{2\pi n}{3}\right)$$

Determine a reasonably simple expression for, and provide a well-labeled plot of, the corresponding output $y(n)$.

The input is easily decomposed into a linear combination of complex exponential functions:

$$x(n) = \underbrace{e^{i(0)n}}_{1} + \underbrace{e^{i(\pi)n}}_{(-1)^n} + \underbrace{\frac{1}{2}e^{-i(2\pi/3)n} + \frac{1}{2}e^{i(2\pi/3)n}}_{\cos(2\pi n/3)} \tag{10}$$

From the eigenfunction property of LTI systems,

$$e^{i\omega n} \to F(\omega)\, e^{i\omega n} \tag{11}$$

It is apparent the filter will "kill" the frequencies at $-2\pi/3 = 4\pi/3$ and $2\pi/3$ by looking at our magnitude plot in (b). We can verify this is the case algebraically.

$$e^{i(-2\pi/3)n} \to F\left(-\frac{2\pi}{3}\right) e^{i(-2\pi/3)n} = \left(1 + 2\cos\left(-\frac{2\pi}{3}\right)\right) e^{-i(2\pi/3)} e^{i(-2\pi/3)n} \tag{12}$$

$$= \left(1 + 2 \times -\frac{1}{2}\right) e^{-i(2\pi/3)} e^{i(-2\pi/3)n} \tag{13}$$

$$= 0 \tag{14}$$

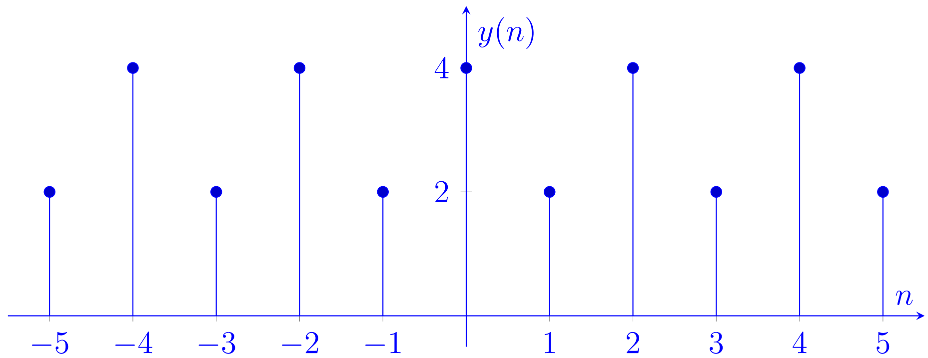
Likewise for $e^{i(2\pi/3)n}$.

Applying the frequency response to the other frequencies,

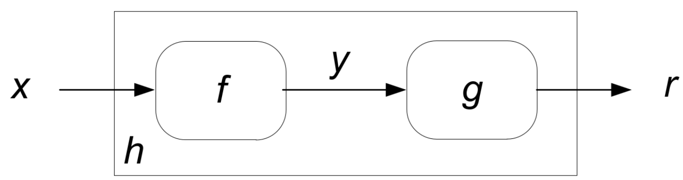
$$e^{i(0)n} \to A(0)\, e^{-i(0)} e^{i(0)n} = (1 + 2\cos(0)) = 3 \tag{15}$$

$$e^{i(\pi)n} \to A(\pi)\, e^{-i\pi} e^{i(\pi)n} = (1 + 2\cos(\pi))(-1)\, e^{i\pi n} = -(1-2)(-1)^n = (-1)^n \tag{16}$$

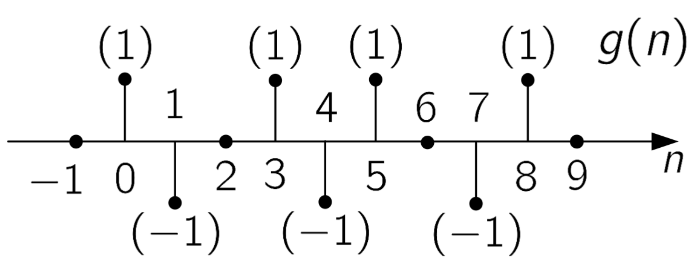
$$y(n) = 3 + (-1)^n \tag{17}$$



(d) (30 points) We place the filter F in cascade (series) with another DT-LTI filter G:



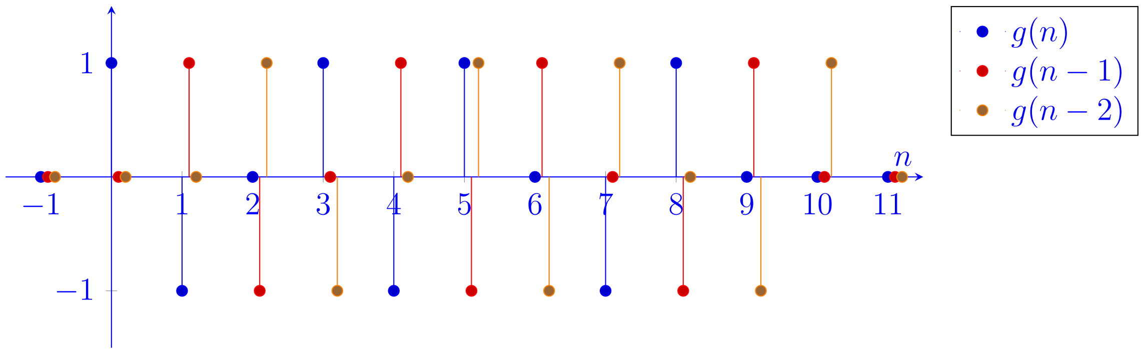
All the nonzero values of the impulse response of system G appear below:



(i) (10 points) Show that the impulse response of the overall system H is

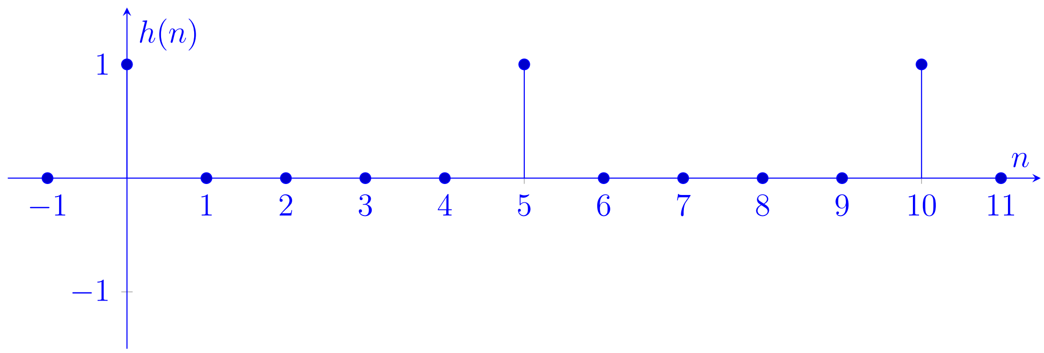
$$\forall n \in \mathbb{Z}, \quad h(n) = \delta(n) + \delta(n-5) + \delta(n-10)$$

$$
\begin{aligned}
y(n) &= (f * x)(n) && (18)\\
r(n) &= (g * y)(n) = (g * (f * x))(n) = ((g * f) * x)(n) && (19)\\
h(n) &= (f * g)(n) && (20)\\
&= \delta(n) * g(n) + \delta(n-1) * g(n) + \delta(n-2) * g(n) && (21)\\
&= g(n) + g(n-1) + g(n-2) && (22)
\end{aligned}
$$



(In the figure above, we applied small offsets to the red and orange signals so they do not completely overlap with each other and the blue signal.)

Summing at each sample, we get:



As desired, the corresponding expression is:

$$h(n) = \delta(n) + \delta(n-5) + \delta(n-10) \tag{23}$$

(ii) (5 points) Determine a reasonably simple expression for the output $r(n)$ if the input is

$$\forall n \in \mathbb{Z}, \quad x(n) = 1 + \cos\left(\frac{\pi n}{5}\right) + \sin\left(\frac{2\pi n}{15}\right)$$

$$\begin{align}
y(n) &= (h * x)(n) \tag{24}\\
&= \delta(n) * x(n) + \delta(n-5) * x(n) + \delta(n-10) * x(n) \tag{25}\\
&= x(n) + x(n-5) + x(n-10) \tag{26}\\
x(n-5) &= 1 + \cos\left(\frac{\pi(n-5)}{5}\right) + \sin\left(\frac{2\pi(n-5)}{15}\right) \tag{27}\\
&= 1 + \cos\left(\frac{\pi n}{5} - \pi\right) + \sin\left(\frac{2\pi n}{15} - \frac{\pi}{3}\right) \tag{28}\\
&= 1 - \cos\left(\frac{\pi n}{5}\right) + \sin\left(\frac{2\pi n}{15} - \frac{2\pi}{3}\right) \tag{29}
\end{align}$$

$$x(n-10) = 1 + \cos\left(\frac{\pi(n-10)}{5}\right) + \sin\left(\frac{2\pi(n-10)}{15}\right) \tag{30}$$
$$= 1 + \cos\left(\frac{\pi n}{5}\right) + \sin\left(\frac{2\pi n}{15} - \frac{4\pi}{3}\right) \tag{31}$$
$$= 1 + \cos\left(\frac{\pi n}{5}\right) - \sin\left(\frac{2\pi n}{15} - \frac{\pi}{3}\right) \tag{32}$$
$$y(n) = x(n) + x(n-5) + x(n-10) \tag{33}$$
$$= 3 + \cos\left(\frac{\pi n}{5}\right) \tag{34}$$

We used the fact that $\sin(x-\pi) = -\sin(x)$—likewise for $\cos(\cdot)$—to cancel one pair of $\cos(\cdot)$ terms, and another pair of $\sin(\cdot)$ terms.

(iii) (15 points) We can factor the polynomial $p(z) = z^{10} + z^5 + 1$ as $p(z) = a(z)\, b(z)$, where

$$a(z) = a_8 z^8 + a_7 b^7 + \cdots + a_1 z + a_0 \quad b(z) = b_2 z^2 + b_1 z + b_0$$

Determine the factorization polynomials $a(z)$ and $b(z)$ explicitly (by determining their respective coefficients numerically).

All the coefficients are integers. Don't do anything wild. Think first before you tackle this part. It should not involve much work (despite the large space we've given you below). The result should be stunningly beautiful, as should the process of getting there.

$a(z)$, $b(z)$, and $p(z)$ resemble $g(n)$, $f(n)$, and $h(n)$, respectively. In particular, we "encoded" each sample of amplitude a with delay k as a monomial az^k.

If we write out the frequency responses of $f(n)$ and $g(n)$, respectively,

$$F(\omega) = \sum_{k=-\infty}^{\infty} f(k)\, e^{-i\omega k} = f(0)\, e^{-i\omega(0)} + f(1)\, e^{-i\omega(1)} + f(2)\, e^{-i\omega(2)} \tag{35}$$
$$G(\omega) = \sum_{k=-\infty}^{\infty} g(k)\, e^{-i\omega k} = g(0)\, e^{-i\omega(0)} + g(1)\, e^{-i\omega(1)} + \cdots + f(8)\, e^{-i\omega(8)} \tag{36}$$
$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-i\omega k} = h(0)\, e^{-i\omega(0)} + h(5)\, e^{-i\omega(5)} + h(10)\, e^{-i\omega(10)} \tag{37}$$

Now, call $e^{-i\omega} = z$, so we can rewrite $F(\omega)$ and $G(\omega)$ as

$$F(\omega) = f(0)\, z^0 + f(1)\, z^1 + f(2)\, z^2 \tag{38}$$
$$G(\omega) = g(0)\, z^0 + g(1)\, z^1 + \cdots + g(8)\, z^8 \tag{39}$$
$$H(\omega) = h(0)\, z^0 + h(5)\, z^5 + h(10)\, z^{10} \tag{40}$$

Then, the overall frequency response will be

$$
\begin{aligned}
H(\omega) &= F(\omega)\, G(\omega) && (41)\\
&= \underbrace{\left(f(0)\, z^0 + f(1)\, z^1 + f(2)\, z^2\right)}_{b(z)} \underbrace{\left(g(0)\, z^0 + g(1)\, z^1 + \cdots + g(8)\, z^8\right)}_{a(z)} && (42)\\
&= p(z) && (43)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
b_0 = b_1 = b_2 &= 1 && (44)\\
a_0 = a_3 = a_5 = a_8 &= 1 && (45)\\
a_1 = a_4 = a_7 &= -1 && (46)\\
a_2 = a_6 &= 0 && (47)
\end{aligned}
$$

Problem 2 (40 points) Consider the DT-LTI filter F whose input-output behavior is described by

$$\forall n \in \mathbb{Z}, \quad y(n) = x(n) - 0.9x(n-1)$$

(a) (10 points) Determine and provide a well-labeled plot of $f(n)$, the impulse response of the filter.

Setting $x(n) = \delta(n)$,

$$
\begin{aligned}
f(0) &= \delta(0) - 0.9\delta(-1) = 1 && (48)\\
f(1) &= \delta(1) - 0.9\delta(0) = -0.9 && (49)
\end{aligned}
$$

For all other $n \notin \{0, 1\}$, the argument to both $\delta(\cdot)$ will be nonzero, so the impulse response there will be zero. Therefore,

$$f(n) = \delta(n) - 0.9\delta(n-1) \tag{50}$$

(b) (5 points) Is the system BIBO stable? Provide a succinct, yet clear and convincing explanation.

Yes—a DT-LTI system is BIBO-stable iff its impulse response is absolutely summable, which it is in this case.

$$\sum_{k=-\infty}^{\infty} |f(k)| = 1 + 0.9 = 1.9 < \infty \tag{51}$$

(c) (10 points) Determine the output of the system in response to the input

$$\forall n \in \mathbb{Z}, \quad x(n) = 0.9^n u(n) + 0.9^n u(-n)$$

$$\begin{aligned} y(n) &= (f * x)(n) && (52)\\ &= \delta(n) * x(n) + 0.1\delta(n-1) * x(n) && (53)\\ &= x(n) - 0.9x(n-1) && (54) \end{aligned}$$

Since $u(0) = u(-0) = 1$, we can rewrite $x(n)$ as $x(n) = 0.9^n + \delta(n)$, so

$$\begin{aligned} y(n) &= 0.9^n + \delta(n) - 0.9 \times 0.9^{n-1} - 0.9 \times \delta(n-1) && (55)\\ &= 0.9^n - 0.9^n + \delta(n) - 0.9\delta(n-1) && (56)\\ &= \delta(n) - 0.9\delta(n-1) && (57) \end{aligned}$$

(d) (10 points) Determine a reasonably simple expression for the frequency response $F(\omega)$, and provide well-labeled plots of the magnitude response $|F(\omega)|$ and phase response $\angle F(\omega)$.

$$F(\omega) = f(0) + f(1)\, e^{-i\omega} = 1 - 0.9e^{-i\omega} \quad (58)$$

While exact closed-form solutions for the magnitude and phase response are somewhat complicated, we can get a pretty good idea about the behavior of the frequency response by analyzing the response graphically in the complex plane.

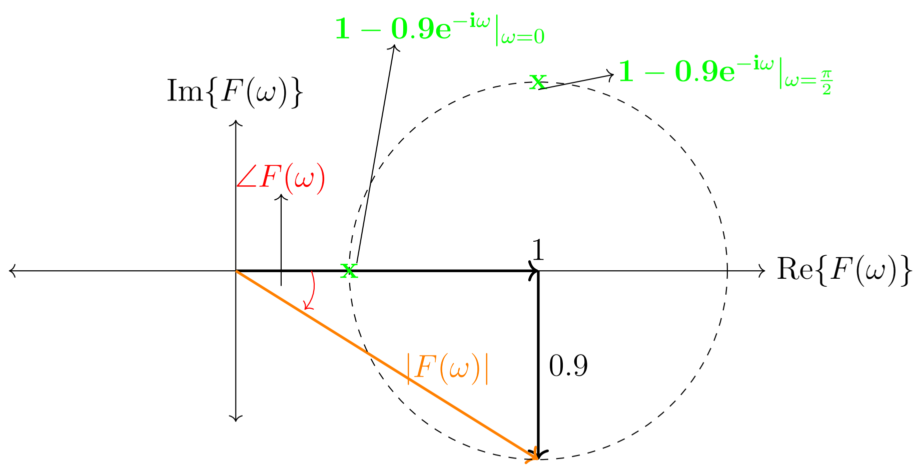
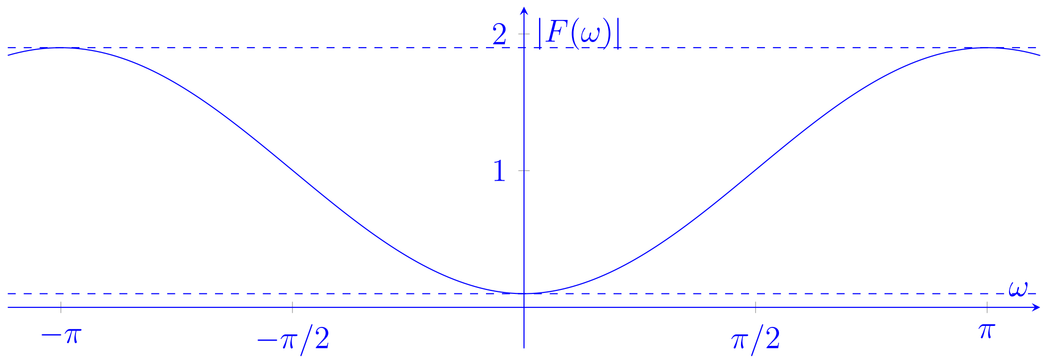
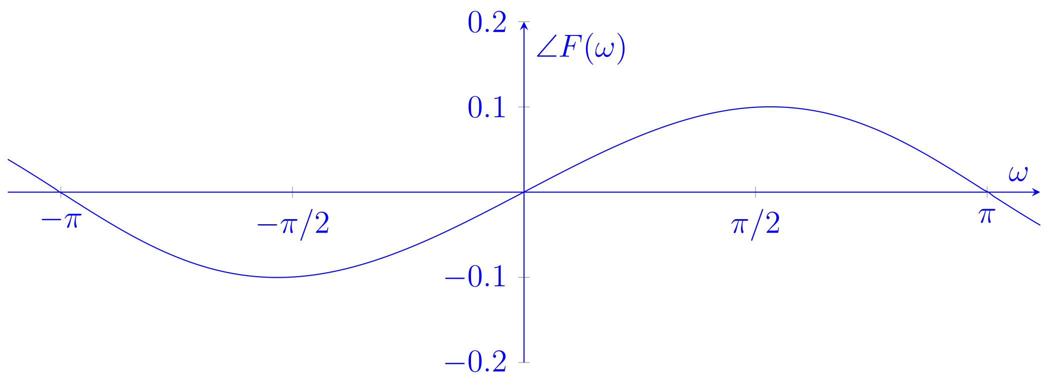


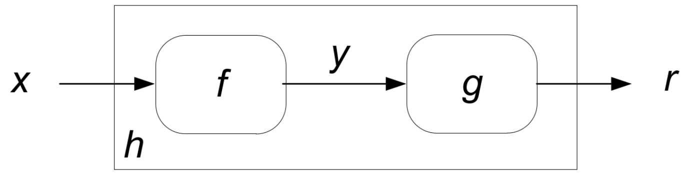
Figure 1: $F(\pi/2)$ (not to scale)

Here we show in green the complex valued function $F(\omega)$ for a couple of values of ω. $\angle F(\omega)$ and $|F(\omega)|$ are shown in orange and red, and in this case correspond to $\omega = \frac{3\pi}{2}$. $|F(\omega)|$ will be at its minimum of 0.1 at $\omega = 0$ and at its maximum of 1.9 at $\omega = \pi$ (this is a highpass filter). $\angle F(0) = \angle F(\pi) = 0$, and the phase will rise above zero slightly on $\omega \in (0, \pi)$ and dip below zero slightly on $\omega \in (0, \pi)$.





(e) (5 points) We place the filter F in cascade (series) with another DT-LTI filter G:



Determine the impulse response $g(n)$ and frequency response $G(\omega)$ of the system G, such that $r(n) = x(n)$. In other words, we want the system G to be the *inverse* of the system F.

If G is the inverse of F, then it should appear that $h(n) = \delta(n)$.

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-i\omega k} = \underbrace{h(0)}_{\delta(0)} e^{-i\omega(0)} = 1 \tag{59}$$

$$H(\omega) = F(\omega)\, G(\omega) \tag{60}$$

$$\implies G(\omega) = \frac{H(\omega)}{F(\omega)} \tag{61}$$

$$= \frac{1}{1 - 0.9e^{-i\omega}} \tag{62}$$

Using pattern matching we can tell that the corresponding impulse response is:

$$g(n) = 0.9^n u(n) \tag{63}$$

There are other ways to achieve $g(n)$ aside from pattern matching. For example, we can use the following approach: Now, we can retrieve difference equation $y(n) \to r(n)$ from the frequency response.

$$G(\omega) - 0.9e^{-i\omega} G(\omega) = 1 \tag{64}$$

$$G(\omega)\, e^{i\omega n} - 0.9e^{i\omega(n-1)} G(\omega) = e^{i\omega n} \tag{65}$$

$$r(n) - 0.9r(n-1) = y(n) \tag{66}$$

G is causal, by inspection, so $g(n) = 0, \forall n < 0$.

$$g(0) - 0.9g(-1) = g(0) = \delta(0) = 1 \tag{67}$$

$$g(1) - 0.9g(0) = \delta(1) = 0 \implies g(1) = 0.9g(0) = 0.9 \tag{68}$$

$$g(2) - 0.9g(1) = \delta(2) = 0 \implies g(2) = 0.9g(1) = 0.9 * 0.9 \tag{69}$$

$$g(n) = (0.9)^n u(n) \tag{70}$$

For another alternative solution, you can note that we are searching for $g(n)$ such that

$$h(n) = f(n) * g(n) = \delta(n) \tag{71}$$

Given the input signal $x(n)$ the intermediate signal $y(n) = x(n) * f(n)$ is the input to g, and so $r(n) = y(n) * g(n)$. Because we design $g(n)$ to be the inverse of $f(n)$, we know that $r(n) = x(n)$.

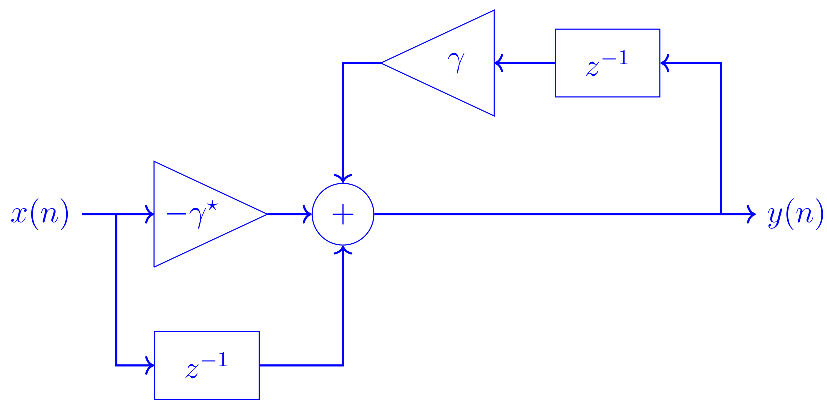
Now, we say that we search for an input $\hat{x}(n)$ such that $\hat{y}(n) = \hat{x}(n) * f(n) = \delta(n)$. This means we have $r(n) = \hat{y}(n) * g(n) = \delta(n) * g(n) = g(n)$. So, in the end, we have $g(n) = r(n) = \hat{x}(n)$. So to find $g(n)$ we must find an input to $f(n)$ that yields $\delta(n)$ as its output.

This can be done by inspection, using intuition about convolution and the flip-and-shift method. Because of the nature of $f(n)$ we look for an input such that subtracting $0.9 * x(n-1)$ from $x(n)$ will yield 0 for all $n \neq 0$. This leads to an exponential decay function, and the unit step function is so that it is not uniformly zero but instead is 1 for $n = 0$.

Problem 3 (40 points) The input-output behavior of a BIBO-stable, discrete-time LTI filter F is described by the lienar, constant-coefficient difference equation

$$y(n) = \gamma y(n-1) - \gamma^\star x(n) + x(n-1), \quad \text{where } |\gamma| < 1$$

(a) (10 points) Provide a delay-adder-gain block diagram implementation of the filter. Your implementation must use the minimum number of delay blocks needed.



Swapping the order of the gain and delay blocks for the $y(n)$ feedback path would have also been fine.

(b) (20 points) Show that the frequency response of the filter is given by

$$\forall \omega \in \mathbb{R}, \quad F(\omega) = \frac{e^{-i\omega} - \gamma^\star}{1 - \gamma e^{-i\omega}},$$

and provide a well-labeled plot of the magnitude response $|F(\omega)|$.

Apply the eigenfunction property ($e^{i\omega n} \to F(\omega)\, e^{i\omega n}$) to the difference equation.

$$F(\omega)\, e^{i\omega n} = \gamma F(\omega)\, e^{i\omega(n-1)} - \gamma^\star e^{i\omega n} + e^{i\omega(n-1)} \tag{72}$$

$$F(\omega) = \gamma F(\omega)\, e^{-i\omega} - \gamma^\star + e^{-i\omega} \tag{73}$$

$$\left(1 - \gamma e^{-i\omega}\right) F(\omega) = \left(e^{-i\omega} - \gamma^\star\right) \tag{74}$$

$$F(\omega) = \frac{e^{-i\omega} - \gamma^\star}{1 - \gamma e^{-i\omega}} \tag{75}$$

$$|F(\omega)| = \left| e^{-i\omega} \frac{1 - \gamma^\star e^{i\omega}}{1 - \gamma e^{-i\omega}} \right| \tag{76}$$

$$= \underbrace{\left| e^{-i\omega} \right|}_{1} \left| \frac{1 - \left(\gamma e^{-i\omega}\right)^\star}{1 - \gamma e^{-i\omega}} \right| \tag{77}$$

We note:

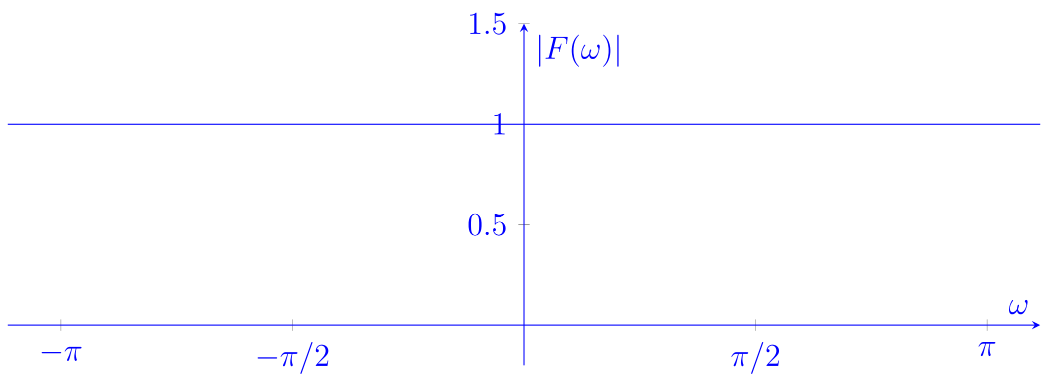
$$1 - \left(\gamma e^{-i\omega}\right)^\star = \left(1 - \gamma e^{-i\omega}\right)^\star \tag{78}$$

And:

$$|\gamma| = |\gamma^\star| \text{ as for } \gamma = a + bi, \gamma^\star = a - bi \rightarrow \sqrt{(a)^2 + (b)^2} = \sqrt{(a)^2 + (-b)^2} \quad (79)$$

F must be an *all-pass filter* (that is, it has unity gain for all frequencies).

$$|F(\omega)| = \frac{\left|(1 - \gamma e^{-i\omega})^\star\right|}{|1 - \gamma e^{-i\omega}|} = 1 \quad (80)$$



(c) (10 points) Determine a reasonably simple expression for $f(n)$, the impulse response of the filter.

By inspection, the system is causal, since the output at time n depends only on previous outputs and inputs at or before n.

$$f(n) = 0, \quad \forall n < 0 \quad (81)$$
$$f(0) = \gamma f(-1) - \gamma^\star \delta(0) + \delta(-1) = -\gamma^\star \quad (82)$$
$$f(1) = \gamma f(0) - \gamma^\star \delta(1) + \delta(0) = -\gamma\gamma^\star + 1 = -|\gamma|^2 + 1 \quad (83)$$
$$f(2) = \gamma \left(1 - |\gamma|^2\right) \quad (84)$$
$$f(n) = \gamma f(n-1) = \gamma^{n-1} \left(1 - |\gamma|^2\right) \quad (85)$$

Problem 4 (50 points) The impulse response of a CT-LTI filter H is given by

$$h(t) = \delta(t) - 2\alpha e^{-\alpha t} u(t)$$

where $\alpha > 0$. In one or more parts of this problem, you may or may not find it useful to know that if the impulse response of a BIBO stable continuous-time LTI system is

$$\beta e^{-\gamma t}, \text{ then its frequency response is } \frac{\beta}{i\omega + \gamma}$$

where γ and β are, in general, complex scalars, with the value of γ consistent with BIBO stability.

(a) (30 points) Determine a reasonably simple expression for $H(\omega)$, the frequency response of the filter, and provide well-labeled plots for the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$.

$$\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\mathrm{d}t && (86)\\
&= \int_{-\infty}^{\infty} \left(\delta(t) - 2\alpha e^{-\alpha t} u(t)\right) e^{-i\omega t}\mathrm{d}t && (87)\\
&= \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\mathrm{d}t - \int_{-\infty}^{\infty} 2\alpha e^{-\alpha t} u(t)\, e^{-i\omega t}\mathrm{d}t && (88)\\
&= \int_{-\infty}^{\infty} e^{-i\omega(0)}\delta(t)\,\mathrm{d}t - 2\alpha \int_{0}^{\infty} e^{-\alpha t} e^{-i\omega t}\mathrm{d}t && (89)\\
&= \int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t - 2\alpha \int_{0}^{\infty} e^{-(\alpha+i\omega)t}\mathrm{d}t && (90)\\
&= 1 + 2\alpha \frac{e^{(-\alpha-i\omega)t}}{\alpha+i\omega}\Big|_{t=0}^{t=\infty} && (91)\\
&= 1 - \frac{2\alpha}{i\omega+\alpha} && (92)\\
&= \frac{i\omega+\alpha}{i\omega+\alpha} - \frac{2\alpha}{i\omega+\alpha} && (93)\\
&= \frac{i\omega-\alpha}{i\omega+\alpha} && (94)\\
&= -\frac{\alpha-i\omega}{\alpha+i\omega} && (95)\\
|H(\omega)| &= \frac{|i\omega-\alpha|}{|i\omega+\alpha|} && (96)\\
&= \frac{\alpha^2+\omega^2}{\alpha^2+\omega_2} && (97)\\
&= 1 && (98)\\
\angle H(\omega) &= \angle(-1) + \angle(\alpha-i\omega) + \angle(\alpha+i\omega) && (99)\\
&= \pi - 2\mathrm{arctan}(\omega/\alpha) && (100)
\end{aligned}$$

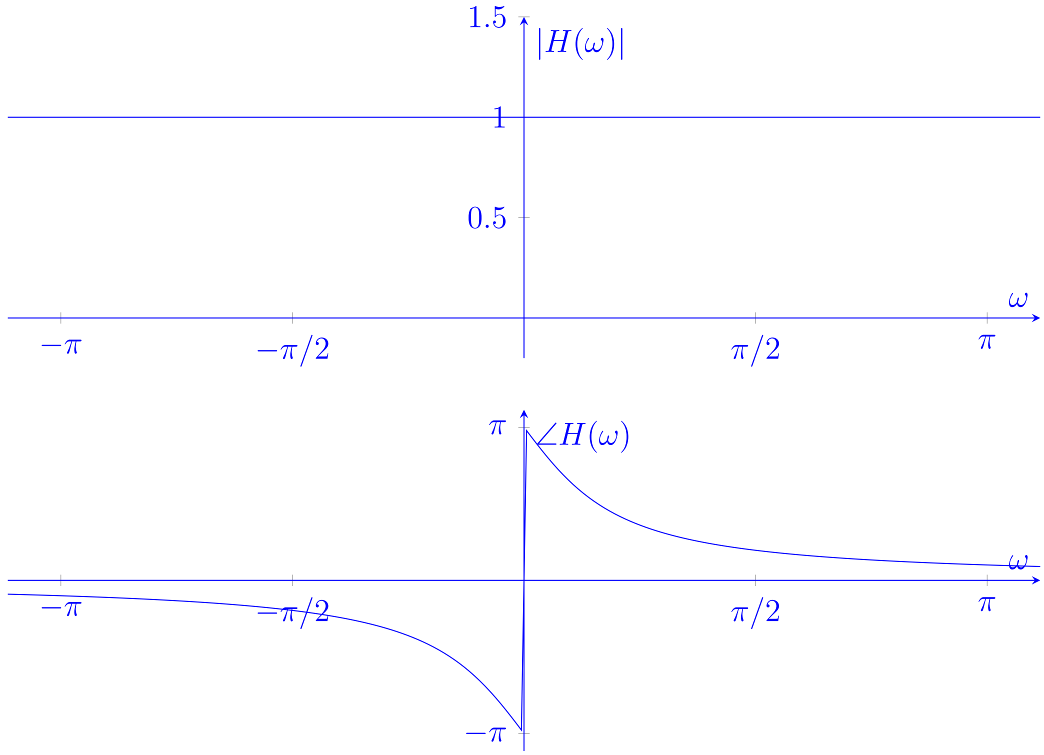


Figure 2: Phase response with $\alpha = 0.5$

(b) (10 points) Determine a reasonably simple expression for, and provide a well-labeled plot of, the filter's step response $s(t)$. Recall that if the input is $x(t) = u(t)$, the corresponding output is $y(t) = s(t)$ and is called the *unit-step response*.

$$
\begin{aligned}
s(t) &= (h * u)(t) && (101)\\
&= \int_{-\infty}^{\infty} \delta(\tau)\, u(t-\tau)\, \mathrm{d}\tau - 2\alpha \int_{-\infty}^{\infty} e^{-\alpha(t-\tau)} u(t-\tau)\, u(\tau)\, \mathrm{d}\tau && (102)\\
&= u(t) - 2\alpha \int_{0}^{\infty} e^{-\alpha(t-\tau)} u(t-\tau)\, \mathrm{d}\tau && (103)
\end{aligned}
$$

Let $x \triangleq t - \tau \implies \mathrm{d}x = -\mathrm{d}\tau$.

$$
\begin{aligned}
s(t) &= u(t) + 2\alpha \int_{x(0)}^{x(\infty)} e^{-\alpha x} u(x)\, \mathrm{d}x && (104)\\
&= u(t) + 2\alpha \int_{t}^{-\infty} e^{-\alpha x} u(x)\, \mathrm{d}x && (105)\\
&= u(t) - 2\alpha \int_{0}^{t} e^{-\alpha x} \mathrm{d}x && (106)\\
&= u(t) + 2e^{-\alpha x}\Big|_{0}^{t} && (107)
\end{aligned}
$$

$$= u(t) + 2\left(e^{-\alpha t} - e^{-\alpha(0)}\right) \tag{108}$$
$$= u(t) + 2\left(e^{-\alpha t} - 1\right) u(t) \tag{109}$$
$$= (2e^{-\alpha t} - 1)u(t) \tag{110}$$

(c) (10 points) Show that the input-output behavior of the filter is described by the linear, constant-coefficient differential equation

$$\dot{y}(t) + \alpha y(t) = \dot{x}(t) - \alpha x(t)$$

Solution 1 Let $x(t) = e^{i\omega t}$. Our LCCDE then becomes:

$$i\omega H(\omega)e^{i\omega t} + \alpha H(\omega)e^{i\omega t} = i\omega e^{i\omega t} - \alpha e^{i\omega t} \tag{111}$$
$$\Rightarrow H(\omega)e^{i\omega t}(i\omega + \alpha) = e^{i\omega t}(i\omega - \alpha) \tag{112}$$
$$\Rightarrow H(\omega) = \frac{(i\omega - \alpha)}{(i\omega + \alpha)} \tag{113}$$

Which matches the frequency response of the system

Solution 2 Let $x(t)$ denote a generic input to the system.

$$y(t) = (h * x)(t) \tag{114}$$
$$= \left(\delta(t) - 2\alpha e^{-\alpha t}u(t)\right) * x(t) \tag{115}$$
$$= x(t) - 2\alpha \int_{-\infty}^{\infty} e^{-\alpha\tau} u(\tau)\, x(t-\tau)\, \mathrm{d}\tau \tag{116}$$
$$= x(t) - 2\alpha \int_{0}^{\infty} e^{-\alpha\tau} u(t-\tau)\, \mathrm{d}\tau \tag{117}$$

Let $u \triangleq t - \tau \implies \mathrm{d}u = -\mathrm{d}\tau$.

$$y(t) = x(t) + 2\alpha \int_{u(0)}^{u(\infty)} e^{-\alpha(t-u)} x(u)\, \mathrm{d}u \tag{118}$$
$$= x(t) + 2\alpha e^{-\alpha t} \int_{t}^{-\infty} e^{\alpha u} x(u)\, \mathrm{d}u \tag{119}$$
$$= x(t) - 2\alpha e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha u} x(u)\, \mathrm{d}u \tag{120}$$
$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} - 2\alpha\left(-\alpha e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha u} x(u)\, \mathrm{d}u + e^{-\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} e^{\alpha u} x(u)\, \mathrm{d}u\right) \tag{121}$$
$$= \frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2\alpha^2 e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha u} x(u)\, \mathrm{d}u - \underbrace{2\alpha e^{-\alpha t} e^{\alpha t} x(t)}_{2\alpha x(t)} \tag{122}$$

$$
\begin{aligned}
\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \alpha y(t) &= \frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2\alpha^2 e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha u} x(u)\,\mathrm{d}u - 2\alpha x(t) && (123) \\
&\quad + \alpha x(t) - 2\alpha^2 e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha u} x(u)\,\mathrm{d}u && (124) \\
\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \alpha y(t) &= \frac{\mathrm{d}x(t)}{\mathrm{d}t} - \alpha x(t) && (125)
\end{aligned}
$$