Problem 1

(a) We consider a pillbox Gaussian surface that has faces in the region to the left and to the right of the charge configuration. By symmetry, any electric field that exists must be normal to the faces. Thus Gauss' law gives

$$2EA = \frac{\sigma A + \rho_E Ad - \rho_E Ad}{\epsilon_0} \tag{1}$$

$$E = \frac{\sigma}{2\epsilon_0} \tag{2}$$

Thus to the left of the object, we have

$$\vec{E} = -\hat{x}\frac{\sigma}{2\epsilon_0} \tag{3}$$

(b) By the same logic as the previous part, we have

$$\vec{E} = \hat{x}\frac{\sigma}{2\epsilon_0} \tag{4}$$

(c) Now we consider a pillbox surface that extends a distance $|x| < d$ into the slab on the left on one end and extends to the right of the rightmost slab on the other end. Then Gauss' law gives

$$EA + \frac{\sigma A}{2\epsilon_0} = \frac{\sigma A + \rho_E Ad - \rho_E A|x|}{\epsilon_0} \tag{5}$$

$$\vec{E} = -\hat{x}\frac{\sigma/2 + \rho_E d - \rho_E|x|}{\epsilon_0} \tag{6}$$

(d) We proceed in a similar fashion and find

$$EA + \frac{\sigma A}{2\epsilon_0} = \frac{\sigma A + \rho_E Ax - \rho_E Ad}{\epsilon_0} \tag{7}$$

$$\vec{E} = \hat{x}\frac{\sigma/2 + \rho_E x - \rho_E d}{\epsilon_0} \tag{8}$$

$$\tag{9}$$

Problem 2

(a) We first consider a ring of charge Q. Along the axis through the center of the ring that is perpendicular to the plane of the ring. For a ring of radius R and a point on the axis a distance x away from the center of the ring, we have

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{R^2 + x^2}} \tag{10}$$

To get the potential on the symmetry axis of the annulus, we break the annulus into infinitesimal rings of charge dq and radius r. Thus we have

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\sqrt{x^2+r^2}} = \frac{2\pi k}{4\pi\epsilon_0}\int_a^b \frac{1}{\sqrt{x^2+r^2}}dr = \frac{k}{2\epsilon_0}\ln\left(\frac{b+\sqrt{b^2+x^2}}{a+\sqrt{a^2+x^2}}\right) \tag{11}$$

where we have used $dq = \sigma dA = \sigma 2\pi r dr = \frac{k}{r}\cdot 2\pi r dr$.

(b) There is only an electric field in the x direction, which we find with

$$\vec{E} = -\nabla V = \frac{k}{2\epsilon_0}\hat{x}\left(\frac{x}{\sqrt{x^2+b^2}(b+\sqrt{b^2+x^2})} - \frac{x}{\sqrt{x^2+a^2}(a+\sqrt{a^2+x^2})}\right) \tag{12}$$

(c) We have $\vec{F} = m\vec{a} = q\vec{E}$. Thus

$$\vec{a} = \frac{q}{m}\vec{E} = \frac{q}{m}\frac{k}{2\epsilon_0}\hat{x}\left(\frac{x}{\sqrt{x^2+b^2}(b+\sqrt{b^2+x^2})} - \frac{x}{\sqrt{x^2+a^2}(a+\sqrt{a^2+x^2})}\right) \tag{13}$$

Problem 3

The capacitor can be thought of as a collection of capacitors that have infinitesimal areas and are in parallel. The distance x between the plates of these infinitesimal capacitors is a function of y, the distance from the bottom of the capacitor. In particular, we have

$$x = d + y\tan(\theta) \tag{14}$$

Then the capacitance of one of the tiny capacitors located at a height y is

$$dC = \epsilon_0\frac{dA}{d+y\tan(\theta)} = \epsilon_0\frac{\sqrt{A}dy}{d+y\tan(\theta)} \tag{15}$$

where we have used $dA = \sqrt{A}dy$. Since all the capacitors are in parallel, we can simply sum up their capacitances via an integral

$$C = \int_0^{\sqrt{A}} \epsilon_0\frac{\sqrt{A}dy}{d+y\tan(\theta)} = \frac{\epsilon_0\sqrt{A}}{\tan(\theta)}\ln(d+\tan(\theta))\Big|_0^{\sqrt{A}} \tag{16}$$

$$= \frac{\epsilon_0\sqrt{A}}{\tan(\theta)}\ln\left(1+\frac{\tan(\theta)\sqrt{A}}{d}\right) \tag{17}$$

Problem 4

(a) Breaking the wire into segiments of length dx, we find the total resistance as

$$R = \int_0^L \frac{\rho dx}{A} = \frac{\rho_0 L}{A}\left(1 - e^{-1}\right) \tag{18}$$

(b) According to Ohm's law

$$j = \frac{E}{\rho} \tag{19}$$

Since the current and area are constant in every cross section of our wire and the resistivity is a decreasing function of x, E must also be a decreasing function of x.

Problem 5

(a) If we look at the free body diagram of the charge on the left, we have

$$\sum F_x = 0 = T_1 \sin(\alpha) - F_e \rightarrow F_e = T_1 \sin(\alpha) \tag{20}$$

$$\sum F_y = 0 = T_1 \cos(\alpha) - mg \rightarrow mg = T_1 \cos(\alpha) \tag{21}$$

For the charge on the right,

$$\sum F_x = 0 = -T_2 \sin(\beta) + F_e \rightarrow F_e = T_2 \sin(\beta) \tag{22}$$

$$\sum F_y = 0 = T_2 \cos(\beta) - mg \rightarrow mg = T_2 \cos(\beta) \tag{23}$$

If we take the ratios of the equations from each mass, we find

$$\tan(\alpha) = \tan\beta = \frac{F_e}{mg} \tag{24}$$

Thus $\alpha = \beta$

(b) When a dipole is immersed in a constant electric field, it will feel a torque given by

$$\vec{\tau} = \vec{p} \times \vec{E} \tag{25}$$

Since the electric field is parallel to $\vec{P}_1$ and antiparallel to $\vec{P}_2$, we have

$$\vec{P}_1 \times \vec{E}_0 = 0 \tag{26}$$

$$\vec{P}_2 \times \vec{E}_0 = 0 \tag{27}$$

$$\tag{28}$$

Thus when we turn on the electric field, nothing happens. The maximum torque that the field can exert corresponds to when the dipoles and field are perpendiular to each other. Thus the maximum torque on each dipole is

$$\tau_{1M} = P_1 E_0 \tag{29}$$

$$\tau_{2M} = P_2 E_0 \tag{30}$$