EE 120: Signals & Systems
Department of Electrical Engineering and Computer Sciences — Midterm Exam #2
University of California, Berkeley — April 11, 2017, 2:10-3:55pm

Instructions:

- There are **four** questions on this midterm, and one extra credit question. Answer each question in the space provided, and **clearly label the parts of your answer**. You can use the additional blank pages at the end for scratch paper if necessary. **Do NOT write answers on the back of any sheet or in the additional blank pages. Any such writing will NOT be graded.**
- Each problem is worth 20 points, and you may solve the problems in any order. The extra credit problem is worth 5 points.
- Show all work. If you are asked to prove something specific, you must give a derivation and not quote a fact from your notes sheet. Otherwise, you may freely use facts and properties derived in class; just be clear about what you are doing!
- None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.
- You may use one double-sided sheet of notes. **No calculators are allowed** (or needed).

Your Name: **Solutions**

Your Student ID:

Name of Student on Your Left:

Name of Student on Your Right:

For official use – do not write below this line!

Q1	**Q2**	**Q3**	**Q4**	**EC**	**Total**

Problem 1. Consider the system shown in Figure 2 below, which is intended for discrete-time processing of a continuous-time signal.

Figure 1: System.

The discrete-time LTI system in the above figure is characterized by the difference equation

$$y[n] = \frac{1}{4}y[n-2] + x[n] - \frac{1}{2}x[n-1].$$

The input $x(t)$ is bandlimited to the interval $|\omega| < B$ rad/sec. The ideal ADC samples the input with the sampling interval $T_s = 0.25$ sec. The ideal DAC also assumes that the sampling interval is T_s.

a) What is the Nyquist rate in Hz? For what values of B will aliasing be avoided?

Solution We have that $\omega_{\max} = B$. Therefore, $f_{\max} = \frac{B}{2\pi}$ Hz and the Nyquist rate is $2f_{\max} = \frac{B}{\pi}$ Hz. We are given that $T_s = 0.25$ sec, which means that $f_s = 4$ Hz. To avoid aliasing, we need

$$f_s = 4 > \frac{B}{\pi} = 2f_{\max} \;\Rightarrow\; B < 4\pi.$$

b) Find the frequency response of the discrete-time LTI system that takes input $x[n]$ to output $y[n]$. What kind of filter (low-pass, high-pass, or band-pass) best describes this system?

Solution From the difference equation, we have

$$H(e^{j\omega}) = \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j2\omega}} = \frac{1}{1+\frac{1}{2}e^{-j\omega}}.$$

Plugging in different values of ω, we see that the discrete-time LTI system characterizes a high-pass filter.

c) Assume that B is chosen such that no aliasing occurs. What is the effective frequency response, i.e., the frequency response from the input $x(t)$ to the output $y(t)$, of the entire system shown in Figure 2?

Solution Observe that the sampling rate is above the Nyquist rate. Therefore, the effective frequency response of the system is

$$H_{\text{eff}}(j\omega) = \begin{cases} \frac{1}{1+\frac{1}{2}e^{-j\omega T_s}} = \frac{1}{1+\frac{1}{2}e^{-j\frac{\omega}{4}}}, & \text{if } |\omega| < 4\pi \\ 0, & \text{otherwise} \end{cases}.$$

d) If we have the input $x(t) = 2\cos(3\pi/2t)$, then the output will be of the form $y(t) = A\cos(\omega_0 t + \theta)$. Find the values of A and ω_0 (assume the ideal ADC does not contain an anti-aliasing filter).

Solution Observe that no aliasing occurs. We have

$$x(t) = 2\cos(\frac{3\pi}{2}t) = e^{j\frac{3\pi}{2}t} + e^{-j\frac{3\pi}{2}t},$$

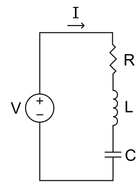
which implies

$$y(t) = H_{\text{eff}}(j\frac{3\pi}{2})e^{j\frac{3\pi}{2}t} + H_{\text{eff}}(-j\frac{3\pi}{2})e^{-j\frac{3\pi}{2}t}.$$

Therefore,

$$A = \left|\frac{1}{1 + \frac{1}{2}e^{-j\frac{3\pi}{8}}}\right| \text{ and } \omega_0 = \frac{3\pi}{2}.$$

Problem 2. Consider the following RLC circuit:



The differential equation relating voltage (V) to the current (I) is given by:

$$\frac{d^2}{dt^2}I(t) + \frac{R}{L}\frac{d}{dt}I(t) + \frac{1}{LC}I(t) = \frac{d}{dt}V(t).$$

Assume we denote the input to the system as $x(t) = V(t)$ and the output of the system as $y(t) = I(t)$. Given that $\frac{R}{L} = 5$ and $\frac{1}{LC} = 4$, we can rewrite the differential equation as

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t).$$

a) Find the transfer function of this system $P(s)$ and identify the ROC (note that real circuits are always causal systems).

Solution

$$P(s) = \frac{s}{s^2+5s+4} = \frac{s}{(s+4)(s+1)} \quad \Re(s) > -1$$

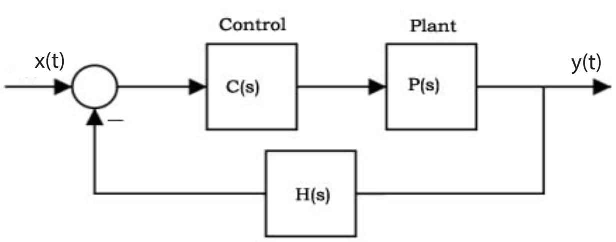
b) Find the impulse response. Is the system stable? Justify your answers.

Solution

$$P(s) = \frac{s}{(s+4)(s+1)} = \frac{\frac{4}{3}}{s+4} + \frac{\frac{-1}{3}}{s+1} \quad \Re(s) > -1 \implies p(t) = (\frac{4}{3}e^{-4t} - \frac{1}{3}e^{-t})u(t)$$

The system is stable since ROC contains s=0 axis.

c) Consider the following feedback diagram:



What is the transfer function of the overall feedback system in terms of $C(s)$, $P(s)$ and $H(s)$?

Solution

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)}$$

d) For the feedback system in part (c), suppose $C(s) = 2$, $P(s)$ is as in part (a), and $H(s) = \frac{7}{4}$. What is the impulse response of the overall system? Is the system stable?

Solution

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)} = \frac{\frac{2s}{s^2+5s+4}}{1 + \frac{2(\frac{7}{4})s}{s^2+5s+4}} = \frac{2s}{s^2 + \frac{17}{2}s + 4} = \frac{2s}{(s+8)(s+\frac{1}{2})}$$

For the ROC $s > -\frac{1}{2}$, the system is stable and causal.

e) For the feedback system in part (c), suppose $C(s) = 2$, $P(s)$ is as in part (a), and $H(s) = \frac{-9}{2}$. Is the resulting system stable?

Solution

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)} = \frac{\frac{2s}{s^2+5s+4}}{1 + \frac{2(\frac{-9}{2})s}{s^2+5s+4}} = \frac{2s}{s^2 - 4s + 4} = \frac{2s}{(s-2)^2}$$

The system is not stable since it has two positive roots at $s = 2$.

Problem 3. A non-ideal sampling operation obtains a discrete-time signal $x_d[n]$ from a continuous-time signal $x(t)$ according to

$$x_d[n] = \int_{nT-T/2}^{nT+T/2} x(t)dt.$$

a) Show that this can be written as ideal sampling of a filtered signal $y(t) = x(t) * h(t)$, that is, $x_d[n] = y(nT)$. Find $h(t)$.

Solution

$$y(t) = \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)d\tau = x(t) * (u(t + \frac{T}{2}) - u(t - \frac{T}{2}))$$

therefore $h(t) = u(t + \frac{T}{2}) - u(t - \frac{T}{2})$.
Note that $H(j\omega) = \frac{2\sin(\omega \frac{T}{2})}{\omega}$ is not band-limited.

b) Express the DTFT of $x_d[n]$ in terms of $X(j\omega)$, $H(j\omega)$ and T.

Solution

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$x_d[n]$ is the sampled version of $y(t)$, therefore:

$$X_d(e^{j\Omega})|_{\Omega=2\pi\frac{\omega}{\omega_s}} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} Y(j(\omega - k\omega_s)) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))H(j(\omega - k\omega_s))$$

where $\omega_s = \frac{2\pi}{T}$.

c) What is the largest B, such that if $x(t)$ is bandlimited to the frequency range $|\omega| < B$, the signal $x(t)$ can be recovered from its samples $x_d[n]$? Is this the same or different from the Nyquist rate?

Solution

$$|\omega| < (\frac{1}{2})\frac{2\pi}{T} \implies B = \frac{\pi}{T}$$

This is equal to the Nyquist rate.

d) Assume that $x(t)$ is bandlimited to the frequency range $|\omega| < 3\pi/(4T)$. Determine the frequency response of a discrete-time system $g[n]$ that will correct the distortion in $x_d[n]$ introduced by the nonideal sampling.

Solution

$$G(e^{j\Omega}) = \frac{1}{H(e^{j\Omega})}$$

Since $\frac{3\pi}{4T} < \frac{\pi}{T}$, there is no aliasing. Therefore:

$$G(e^{j\Omega})|_{\Omega=2\pi\frac{\omega}{\omega_s}} = \frac{1}{H(j\omega)} = \frac{\omega}{2\sin(\frac{\omega T}{2})} \qquad |\omega| < \frac{\omega_s}{2}$$

Problem 4. A causal LTI system with rational transfer function $H(s)$ has poles at $s = -1 \pm j0.5$, and zeros at $s = \pm j1.5$.

a) Plot the pole-zero diagram for this system and shade the ROC. Is the system stable?

Solution The system is stable because the ROC contains the imaginary axis in the s-plane.

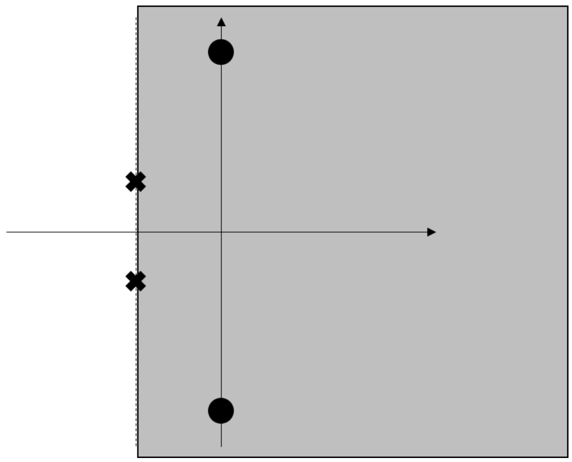


Figure 2: Pole-Zero Diagram and ROC.

b) If the constant DC signal $x(t) = 1$ is input into the system, then it is observed that the signal $y(t) = -1$ is output. Is this enough information to determine $H(s)$? If so, write an explicit expression for $H(s)$.

Solution Yes, this is enough information to determine $H(s)$. We have

$$H(s) = A\frac{(s+j1.5)(s-j1.5)}{(s+1+j0.5)(s+1-j0.5)}.$$

We know that

$$H(0) = A\frac{-1.5^2}{1.25} = -1 \;\Rightarrow\; A = -\frac{5}{9}.$$

$$\therefore\; H(s) = -\frac{5}{9}\frac{(s+j1.5)(s-j1.5)}{(s+1+j0.5)(s+1-j0.5)}.$$

c) What is the output $y(t)$ of this system in response to the input $x(t) = 4+\cos(t/2+\pi/3)$?

Solution We have

$$x(t) = 4 + \frac{1}{2}e^{j\frac{\pi}{3}}e^{j\frac{t}{2}} + \frac{1}{2}e^{-j\frac{\pi}{3}}e^{-j\frac{t}{2}}.$$

Therefore,

$$y(t) = -4 + H(j/2)\frac{e^{j\frac{\pi}{3}}}{2}e^{j\frac{t}{2}} + H(-j/2)\frac{e^{-j\frac{\pi}{3}}}{2}e^{-j\frac{t}{2}}.$$

d) Write a differential equation relating $y(t)$ and $x(t)$ that is consistent with your expression for $H(s)$ in part (b).

Solution Simplifying $H(s)$ in part (b), we obtain

$$H(s) = \frac{Y(s)}{X(s)} = -\frac{5}{9}\frac{s^2 + \frac{9}{4}}{s^2 + 2s + \frac{5}{4}}.$$

$$\therefore\ (9s^2 + 18s + \frac{45}{4})Y(s) = (-5s^2 - \frac{45}{4})X(s),$$

$$\Rightarrow\ 36\frac{d^2y(t)}{dt^2} + 72\frac{dy(t)}{dt} + 45y(t) = -20\frac{d^2x(t)}{dt^2} - 45x(t).$$

e) (Unrelated to previous parts) Solve the following differential equation, assuming initial conditions $y(0^-) = y'(0^-) = 1$,

$$y''(t) - 3y'(t) + 2y(t) = 0.$$

Solution Taking the unilateral Laplace transform, we obtain

$$s^2Y(s) - sy(0^-) - y'(0^-) - 3(sY(s) - y(0^-)) + 2Y(s) = 0.$$

$$\therefore\ (s^2 - 3s + 2)Y(s) = s - 2,$$

$$Y(s) = \frac{s-2}{s^2 - 3s + 2} = \frac{s-2}{(s-1)(s-2)} = \frac{1}{s-1}.$$

Therefore,

$$y(t) = e^t u(t).$$

Problem 5. (5-points Extra Credit) Suppose the human eye samples at a rate of 60 Hz. You observe that the wheels on the car driving next to you appear to be standing perfectly still. If you are driving 65mph ($\approx$29m/s), how fast is the car next to you traveling? (1 m/s $\approx$ 2.2 mph).

Assume the wheels on the car next to you have a circumference of 1.5m, and have 6 spokes (i.e., they look identical regardless of whether they are rotated 0°, 60°, 120°, 180°, 240°, or 300°).

Solution In $\frac{1}{60}$ of a second, I need $\frac{\pi}{3}k$ rad of rotation, where k is a positive integer. In other words, we need an angular frequency that is

$$\frac{\frac{\pi}{3}}{\frac{1}{60}}k = 20\pi k \text{ rad/sec.}$$

$20\pi k$ rad/sec is equivalent to $10k$ revolutions per second. With a wheel circumference of 1.5 m, this means that we need $15k$ m/s. The only k that makes sense in our problem is $k = 2$, so we have that the car next to us is traveling at 30 m/s (approximately 66 mph or 67.2 mph).