EE 120: Signals & Systems
Department of Electrical Engineering and Computer Sciences — Final Exam
University of California, Berkeley — May 8, 2018, 11:30am

Instructions:

- There are **fifteen** questions on this exam, for a grand total of 83 points. **Answer each question in the space provided.** You can use the additional blank pages at the end for scratch paper if necessary.
- The first 10 questions are short answer, they do not require you to show your work. The last 5 questions are longer format; please show all work for these questions. If you are asked to prove something specific, you must give a derivation and not quote a fact from your notes sheet. Otherwise, you may freely use facts and properties derived in class; just be clear about what you are doing!
- We may use Gradescope for grading. **Do NOT write answers on the back of any sheet or in the additional blank pages, it will NOT be scanned or graded.**
- Problems are worth a variable number of points, and you may solve the problems in any order.
- None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.
- You may use two double-sided sheets of notes. **No calculators are allowed** (or needed).

Your Name: TOM
Your Student ID:

Name of Student on Your Left:
Name of Student on Your Right:

For official use – do not write below this line!

Problem 1. (9 points) For each entry in the box, put a 'Yes' or a 'No'. A correct answer gets 1 point, a wrong or blank gets 0 points. A system is described by how it transforms the (discrete or continuous time) input $x(\cdot)$ to the (discrete or continuous time) output $y(\cdot)$. No justification is needed for your answers.

System Description	Linear?	Time-Invariant?	Causal?
$y(t) = x(-t)$	Y	N	N
$y(t) = t^2 x(t-2)$	Y	N	Y
$y[n] = \lvert x[n] \rvert$	N	Y	Y

Problem 2. (2 points) Suppose the input to a continuous time LTI system is $u(t)$, and the observed output is $y(t)$. What is the impulse response $h(t)$ for the system in terms of $y(t)$?

$$\frac{d}{dt} y(t)$$

Problem 3. (2 points) A periodic signal $x(t)$ is input to a continuous time LTI system. Assuming the output is non-constant, is it necessarily periodic?

Yes

Problem 4. (3 points) Suppose we sample a sinusoid $x(t) = \sin(50\pi t)$ at rate ω_s rad$\times$samples/second. Upon passing the samples through an ideal interpolator, the reconstructed continuous time signal is $x_r(t) = \sin(10\pi t)$. What is ω_s? If there is more than one possible answer, give the smallest one.

$\omega_s = 40\pi$

Problem 5. (2 points) A k-point moving average filter has impulse response $h_k[n] = 1/k$ for $0 \leq n \leq k-1$, and $h_k[n] = 0$ otherwise. Does cascading two 6-point moving average filters result in a 12-point moving average?

No

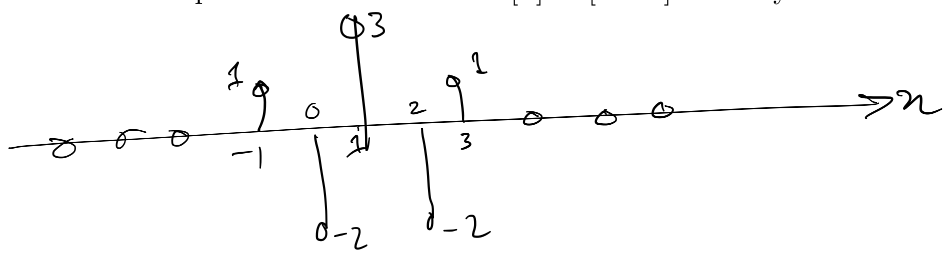
Problem 6. (1 point) Suppose a discrete time periodic signal is input to a stable LTI system. What mathematical tool is **most** suitable for analysis in this situation: DTFT, DTFS, CTFT, CTFS, Laplace transform, or z-transform?

DTFS.

Problem 7. (3 points) Consider the signal

$$x[n] = \begin{cases} (-1)^n & -1 \leq n \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Sketch a plot of the convolution $x[n] * x[n-1]$. Label your axes clearly.



Problem 8. (4 points) Let $x(t)$ be a bandlimited signal with spectrum $X(j\omega) = 0$ for $|\omega| \geq \omega_0$. Determine the Nyquist rate for each of the following signals.

a) $y_1(t) = 3x'(t)$. $2\omega_0$

b) $y_2(t) = (x(t))^4$. $4 \cdot 2\omega_0 = 8\omega_0$.

Problem 9. (2 points) Suppose a system has output $y(t) = 2\sin(\pi t)$ when the input is $x(t) = \sin(\pi t)$. Furthermore, if the input is $x(t) = \sin(\pi t) + \cos(2\pi t)$, the output if the system is $y(t) = 2\sin(\pi t) + 2\cos(2\pi t)$. Based on this information, can you conclude that this a linear system?

No.

Problem 10. (3 points) Consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

What is a **simple** expression for $x(t)$?

Hint: Do not try to do this from scratch. Think of where you have seen a similar sum before. You shouldn't have to do any math to find the answer.

$x(t) = \cos(\pi t)$

Problem 11. (11 points) Consider the following continuous time signal

$$x(t) = 2\sin(200\pi t) \cdot \cos(800\pi t).$$

a) What is the period of $x(t)$?

$x(t) = -\sin(600\pi t) + \sin(1000\pi t)$

$\Rightarrow \quad \omega_0 = 200\pi \quad \Rightarrow \quad T = 1/100 \ s.$

b) Express $x(t)$ in terms of its Fourier Series.

$a_3 = j/2 \ , \quad a_{-3} = -j/2$

$a_5 = -j/2 \ , \quad a_{-5} = j/2 \ .$

c) Suppose now that you observe

$$y(t) = x(t) + z(t),$$

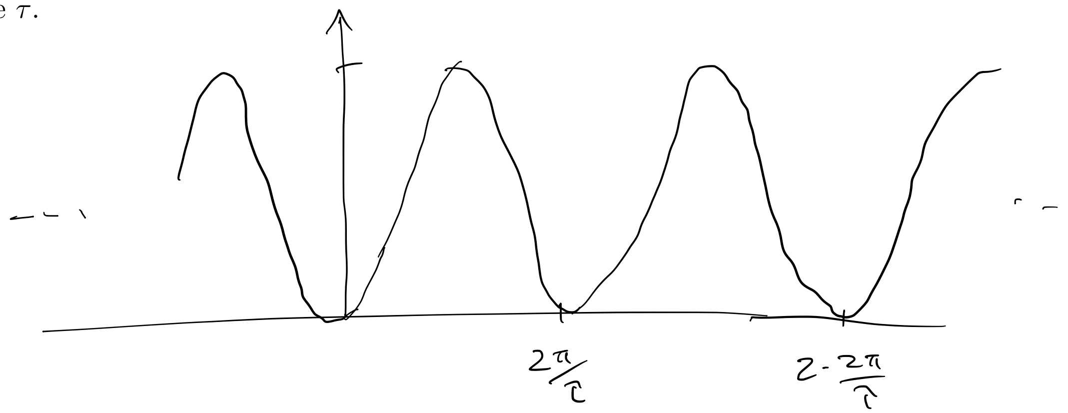
where $z(t) = 20\sin(400\pi t)$. You will attempt to reconstruct $x(t)$ from $y(t)$ via an LTI system $y(t) \mapsto \hat{x}(t)$ with system equation

$$\hat{x}(t) = \frac{1}{2}y(t) - \frac{1}{2}y(t-\tau),$$

where $\tau > 0$ is a design parameter. What is the frequency response $H(j\omega)$ of this system, in terms of τ?

$H(j\omega) = \frac{1}{2}\left(1 - e^{-j\omega\tau}\right)$

d) Plot the gain $|H(j\omega)|^2$ of the system from part (c). Your tick marks on the axes will involve τ.



e) How should you pick τ so that $\hat{x}(t) = x(t)$? If there are multiple such τ's, choose the smallest one.

need to choose τ s.t.

$$H(j600\pi) = H(j1000\pi) = 1$$

$$H(400\pi) = 0$$

picking $\tau = 1/200$ works.

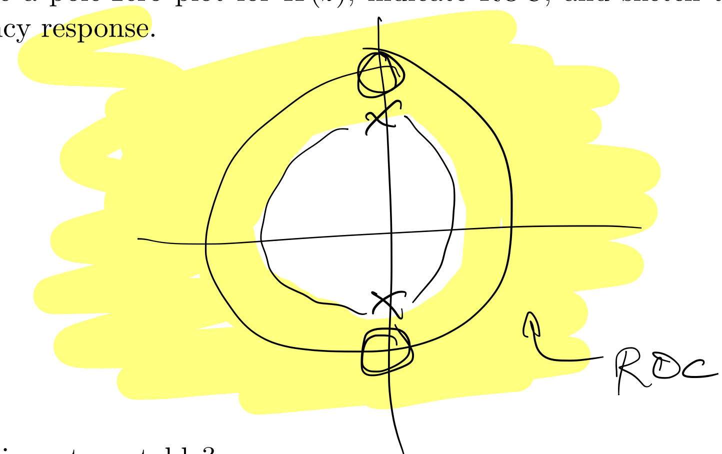
Problem 12. (10 points) Consider the causal LTI system described by the difference equation

$$y[n] + 0.81y[n-2] = x[n] + x[n-2].$$

a) Find the transfer function $H(z)$ for this system.

$$H(z) = \frac{z^2 + 1}{z^2 + 0.81}$$

b) Make a pole-zero plot for $H(z)$, indicate ROC, and sketch the corresponding frequency response.



c) Is this system stable?

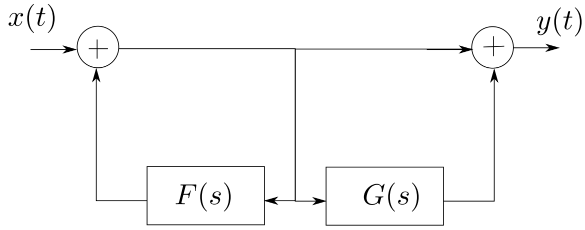
Yes

d) Determine the system output $y[n]$ for the input

$$x[n] = 1 + \sin(\pi n/2).$$

$$y[n] = \frac{2}{1.81}$$

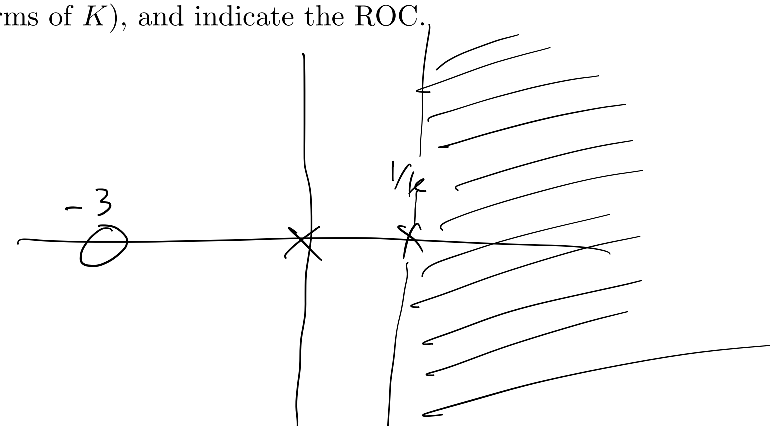
Problem 13. (11 points) Suppose two causal LTI systems with transfer functions $F(s)$ and $G(s)$ are connected as shown below.



a) What is the overall transfer function $H(s)$ for this system, in terms of $F(s)$ and $G(s)$?

$$\frac{1+G(s)}{1-F(s)}$$

b) Suppose $G(s) = 3/s$ and $F(s) = Ks$. Make a pole-zero plot of this system (your plot will be in terms of K), and indicate the ROC.



c) Are there conditions on K for which the system is stable?

No, a zero is always on imag axis.

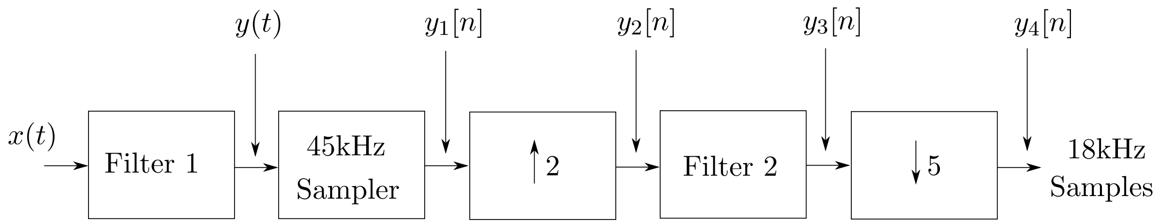
d) Determine the system impulse response $h(t)$.

$$H(s) = \frac{3}{s} + \frac{1+3k}{1-ks}$$

$$= \frac{3}{s} - \frac{3+1/k}{s-1/k}$$

$$\Rightarrow \quad h(t) = 3u(t) - (3+1/k)e^{-1/k\,t}\,u(t) .$$

Problem 14. (10 points) Suppose an audio signal $x(t)$ is sampled at a sampling rate of $f_s = 45\text{kHz}$. However, the device that will ultimately play back these samples assumes the audio signal was sampled at 18kHz. You propose a system like that shown below. The boxes with $\uparrow 2$ and $\downarrow 5$ represent upsampling (i.e., zero-insertion) by a factor of 2, and downsampling (i.e., sample removal) by a factor of 5, respectively.

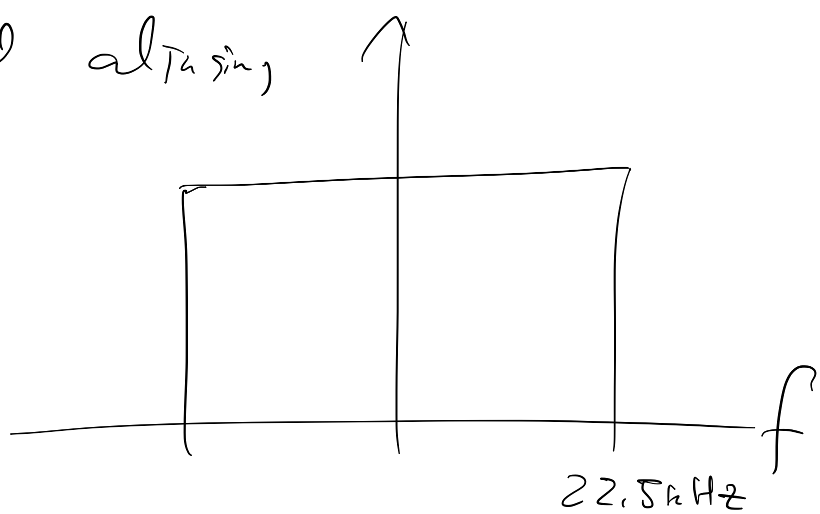


a) State the Sampling Theorem.

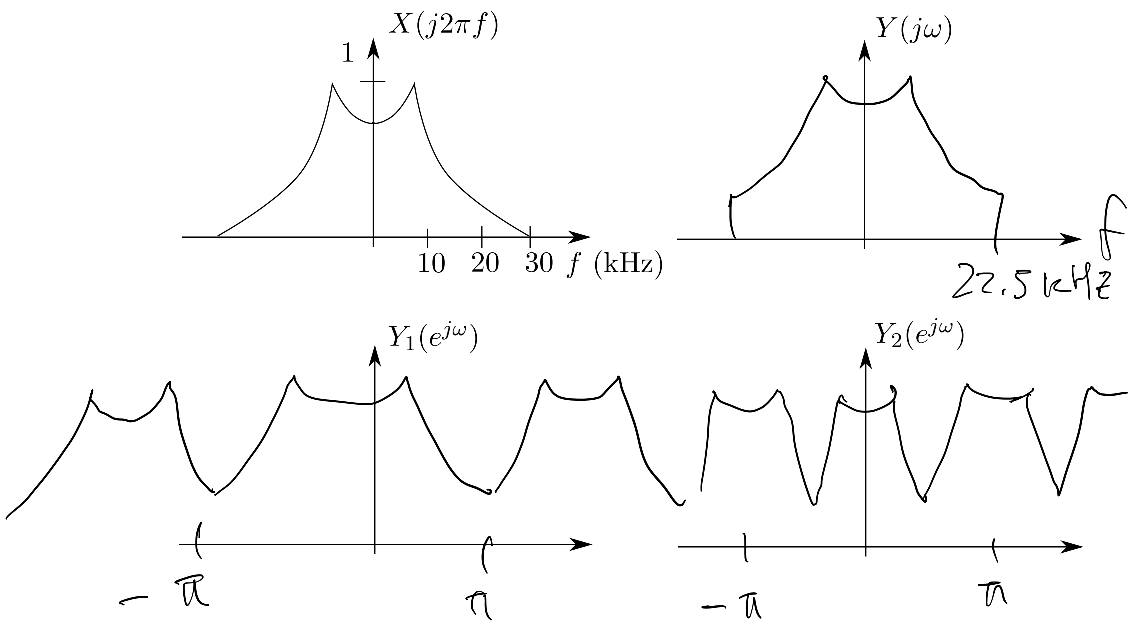
If $x(t)$ bandlimited to ω_M, then sampling at $\omega_s > 2\omega_M$ sufficient to reconstruct losslessly.

b) What is the purpose of Filter 1? Sketch what the ideal frequency response $H_1(j\omega)$ should look like for this filter.

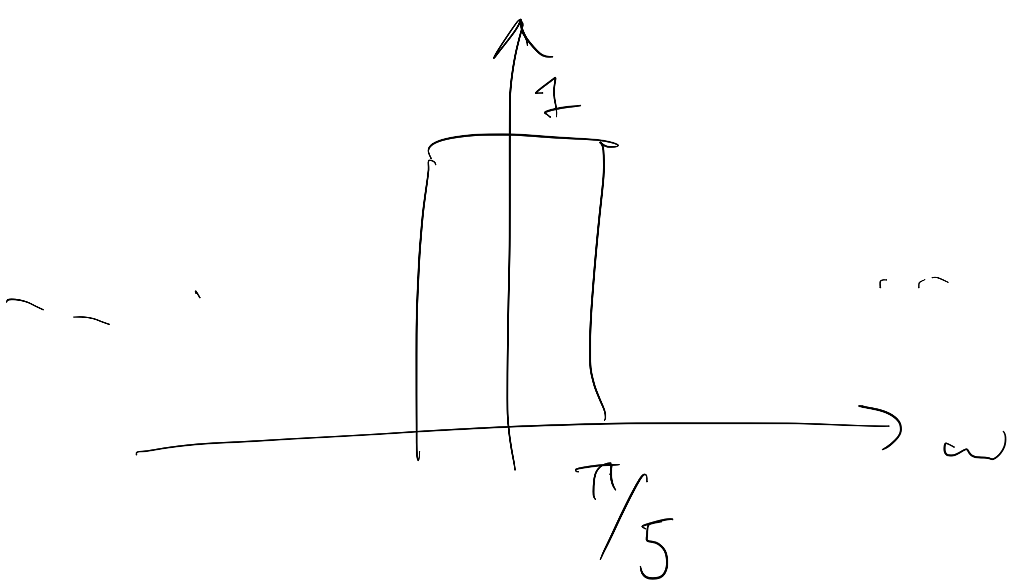
Avoid aliasing



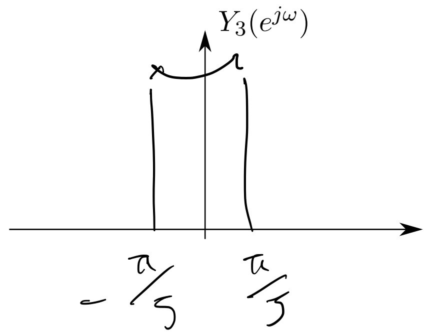
c) Assuming the spectrum of $x(t)$ is as shown below (note the units are in Hz for your convenience), sketch the spectrums for $y(t)$, $y_1[n]$, and $y_2[n]$ on the axes supplied. Assume Filter 1 is as you specified in part (b). Make sure to label the axes.

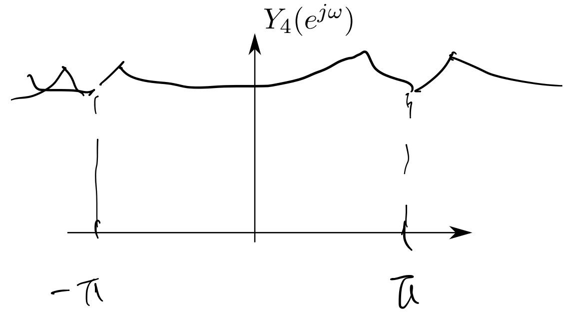


d) Filter 2 is used to prevent aliasing in the downsampling block. Sketch what the ideal frequency response $H_2(e^{j\omega})$ should look like for this filter.



e) Sketch the spectrums for $y_3[n]$ and $y_4[n]$ on the axes supplied. Assume Filter 2 is as you specified in part (d). Make sure to label the axes.





Problem 15. (10 points) In this problem, we establish a Parseval-like theorem for sampled signals.

a) Define $\text{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$. For m, n integers, establish the following identity:

$$\int_{-\infty}^{\infty} \text{sinc}(Wt - m)\,\text{sinc}(Wt - n)dt = \begin{cases} 0 & m \neq n \\ \frac{1}{W} & m = n. \end{cases}$$

Parseval: $\int x(t) y(t) dt = \frac{1}{2\pi} \int X(j\omega) Y^*(j\omega) d\omega$

$$\frac{\sin(\pi W t)}{\pi W t} \longleftrightarrow \begin{cases} 1/W & |\omega| < \pi W \\ 0 & |\omega| > \pi W \end{cases}$$

$$\Rightarrow \text{sinc}(Wt - m) \longleftrightarrow \frac{1}{W} e^{-j\omega m / W} \mathbb{1}\{|\omega| < \pi W\}$$

$$\Rightarrow \text{integral} = \frac{1}{2\pi W^2} \underbrace{\int_{-\pi W}^{\pi W} e^{j\omega(n-m)/W} d\omega}$$

$$= \begin{cases} 2\pi W & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

b) A signal is bandlimited to ω_0 rad/s, and is sampled at $\omega_s > 2\omega_0$. Define $T = 2\pi/\omega_s$ and use the result of part (a) to show that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = T \sum_{n=-\infty}^{\infty} |x(nT)|^2.$$

Hint: Use the fact that $x(t)$ is bandlimited to write $x(t)$ as a sinc interpolation of the samples $x(nT)$, $n \in \mathbb{Z}$.

define $f_s = \frac{1}{T} = \omega_s/2\pi$

$$x(t) = \sum x(nT) \operatorname{sinc}(f_s t - n)$$ by sinc interpolation

$$\Rightarrow |x(t)|^2 = \sum_m \sum_n x(nT) x(mT) \operatorname{sinc}(f_s t - n) \operatorname{sinc}(f_s t - m)$$

integrate both sides, cross-terms cancel by part (a),

so

$$\int |x(t)|^2 dt = \underbrace{f_s}_{= 1/T} \sum_{n \in \mathbb{Z}} |x(nT)|^2$$

(End of Exam)

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