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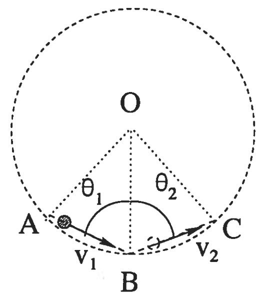
University of California, Berkeley
Mechanical Engineering
ME104 Engineering Mechanics **SOLUTIONS**
Test 2 F17 *Prof S. Morris*

1. (100) A particle P of mass m is tethered to point O and moves on a frictionless horizontal surface. If the length of the tether is constant, P moves in uniform circular motion, but if the tether is shortened, the speed increases. By Newton's second law, that increase in speed requires there to be a component of the tethering force along the particle path. This effect can be understood by approximating the particle path as the sum of rectilinear segments. On each rectilinear segment, no force acts on P; but at corners, such as B, the particle P experiences an impulse of magnitude J directed towards O. Let v_1 be the speed of P on AB; and v_2, that on BC.



(a) Using the impulse– momentum theorem, show that

$$v_2 = v_1 \frac{\sin\theta_1}{\sin\theta_2}, \quad \text{and} \quad J = mv_1 \frac{\sin(\theta_1+\theta_2)}{\sin\theta_2}. \qquad (1.1a, b)$$

(b) On a sketch show the vectors $m\mathbf{v_1}$, $m\mathbf{v_2}$ and $\mathbf{J}$. Using your sketch, explain why $v_2 > v_1$ if $\theta_2 < \theta_1$.

(c) For $\theta_2 = \theta_1$, show that P takes a time $4(r/v_1)\cos\theta_1$ to travel from A to C. By taking the limit as $n \to \infty$, explain why for uniform circular motion, the tethering force $F = mv_1^2/r$.

Solution

The impulse–momentum equation states that $m\mathbf{v_2} - m\mathbf{v_1} = \mathbf{J}$, the impulse $\mathbf{J} = \int \mathbf{F}\,dt$.

(a) By taking the component of that equation perpendicular to BO

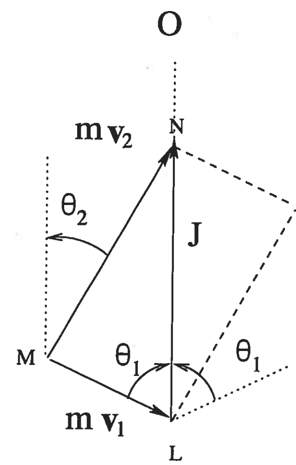
$$\underline{e}_x: \quad mv_2\sin\theta_2 - mv_1\sin\theta_1 = 0; \quad (20\text{ pts}) \qquad (1.2a)$$

and, by taking the component along BO

$$\underline{e}_y: \quad mv_2\cos\theta_2 - m_1\cos\theta_2 = J \quad (20\text{ pts}) \qquad (1.2b)$$

Equation (1.1a) follows by solving (1.2a) for v_2. Equation (1.1b) then folows by substituting for v_2 and using the identity $\sin(A+B) = \sin A\,\cos B + \cos A\,\sin B$.

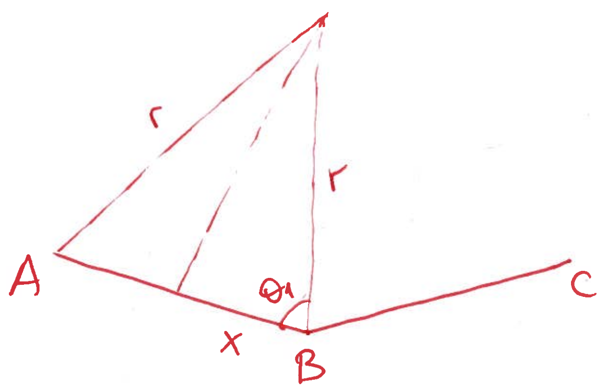
(b) The impulse–momentum equation can be written as $m\mathbf{v_2} = m\mathbf{v_1} + \mathbf{J}$. The sketch shows the effect of decreasing angle θ_2 from the value θ_1 indicated by the broken line: the tip of the vector $\mathbf{J}$ recede from the vertex L of $\triangle LMN$. Because vector $\mathbf{J}$ always lies along side LN pointing towards the centre O, reducing θ_2 will cause $\mathbf{v_2}$ to increase if $\theta_2 < \theta_2 \leq \pi/2$; this is consistent with (1.1a). (10 pts)



(20 pts)

(c) Because an impulse acts at each vertex, the average force experienced by the particle as it travels from A to C is given by the impulse exerted divided by the time needed for P to travel from the midpoint of AB to the midpoint of BC. By geometry, the distance travelled along half the segment AB is $r\cos\theta_1$, so that for $\theta_2 = \theta_1$, the time taken to travel between midpoints of the segments at speed v_1 is $t = 2(r/v_1)\cos\theta_1$. Writing J in (1.1b) in the form Ft where F is the radial force averaged over time t, and setting $\theta_2 = \theta_1$, we have

$$mv_1\frac{\sin 2\theta_1}{\sin\theta_1} = 2\frac{rF}{v_1}\cos\theta_1, \;\Rightarrow\; F = \frac{mv_1^2}{r}\frac{\sin 2\theta_1}{2\cos\theta_1\,\sin\theta_1}, = m\frac{v_1^2}{r}. \tag{1.3}$$



$x = r\cos\theta_1$

$V_1 = \frac{ds}{dt} \Rightarrow V_1 t = 2r\cos\theta_1$

$\Rightarrow t = \frac{2r}{V_1}\cos\theta_1 \quad A \to B$

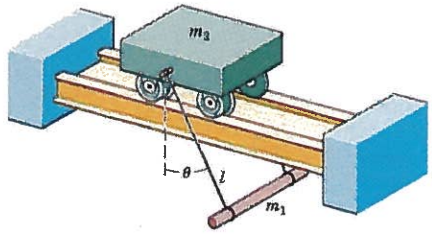
$\Rightarrow 2t = \frac{4r}{V_1}\cos\theta_1 \quad A \to C$

(15 pts)

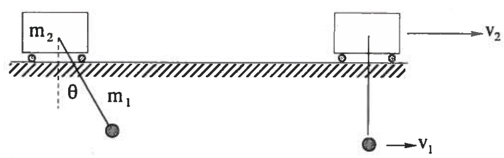
$J = Ft$

$mv_1\frac{\sin(2\theta_1)}{\sin\theta_1} = \left(\frac{2r}{V_1}\cos\theta_1\right)F \to F = \frac{mv_1^2}{r}$ (15 pts)

2. (100) A horizontal bar of mass m_1 and negligible diameter is suspended by two wires of length ℓ from a cart of mass m_2. The cart is free to roll along the horizontal rails. The bar and cart are released from rest with the wire making an angle θ to the vertical. Assuming negligible friction, find the velocity v_2 of the cart, and the velocity $v_{1/2}$ of the bar relative to the cart when $\theta = 0$.



Solution



Because friction is negligible, the mechanical energy of the bar–cart system is conserved. Let the datum for potential energy pass through the lowest point attained by the bar.

Initially, the potential energy is $U = m_1 g\ell\{1 - cos\theta\}$ (positive because m_1 is above the datum); the kinetic energy is zero.

In the final state, the potential energy is zero; the kinetic energy $T = \frac{1}{2}m_2 v_2 + \frac{1}{2}m_1 v_1^2$.

Equating the initial energy to the final energy:

$$\frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_1 v_1^2 + 0 = 0 + m_1 g\ell\{1 - cos\theta\} \qquad (25 \text{ pts}) \qquad (2.1a)$$

This is one equation in two unknowns v_1, v_2.

Because zero external force is exerted on the bar–cart system in the horizontal direction, the horizontal component of momentum is conserved:

$$m_1 v_1 + m_2 v_2 = 0. \qquad (25 \text{ pts}) \qquad (2.1b)$$

This provides the second equation. We note that v_1 in the sketch above is, in fact, negative.

By eliminating the velocity v_1 of the bar between (2.1a) and (2.1b),

$$v_2 = m_1\left[\frac{2g\ell(1 - \cos\theta}{m_2(m_1 + m_2)}\right]^{1/2}. \qquad (2.2)$$

So $v_2 \to 0$ as $m_1/m_2 \to 0$, as we would expect.)

Because $v_1 = v_2 + v_{1/2}$,

$$v_{1/2} = v_1 - v_2, = -\left(\frac{m_2}{m_1} + 1\right)v_2. \qquad (20 \text{ pts}) \qquad (2.3)$$

Equation (2.1b) has been used.

Solving V_2, $V_{1/2}$ → (30 pts)

By substituting for v_2 from (2.2),

$$v_{1/2} = -\left[2g\ell\left(1+\frac{m_1}{m_2}\right)(1-\cos\theta)\right]^{1/2}. \tag{2.4}$$

As $m_1/m_2 \to 0$ (light rod), $v_{1/2} \to [2g\ell(1-\cos\theta)^{1/2}]$, because v_2 then vanishes, and the bar behaves as a simple pendulum.