EE 20N: Structure and Interpretation of Signals and Systems
Department of Electrical Engineering and Computer Sciences — MIDTERM 3
UC BERKELEY — 4 December 2014

FIRST Name Sam LAST Name Pell

Lab Day & Time: 24/7 SID (All Digits): 0123456789

- **(10 Points)** Print your *official* name (not your e-mail address) and *all* digits of your student ID number legibly, and indicate your lab time, on *every* page.
- This exam should take up to 80 minutes to complete. You will be given at least 80 minutes, up to a maximum of 90 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

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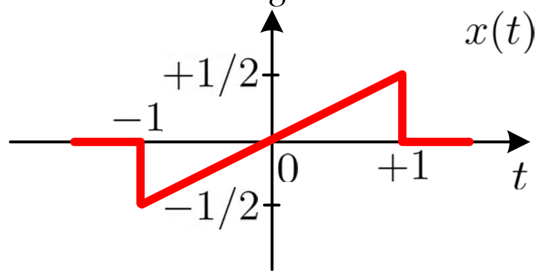
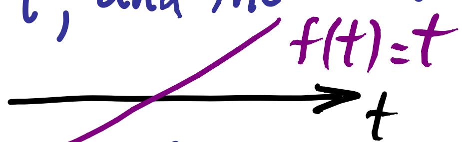
MT3.1 (45 Points) You may tackle parts (a) and (b) of this problem in either order.

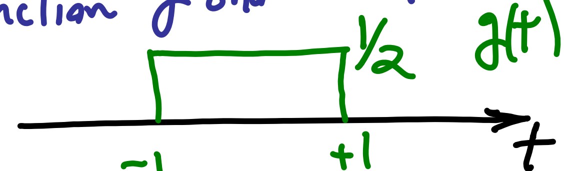
(a) (25 Points) The continuous-time (CT), sawtooth, single-pulse signal x is zero everywhere except as shown in the figure below:



(i) (15 Points) Determine a reasonably simple expression for $X(\omega)$, the spectrum of the signal x.

We can think of x as the product of the signal f described by $f(t) = t$, for all t, and the "box function" g shown below:





That is to say, $x(t) = f(t) g(t)$ for all t.

Clearly, then, we have $x(t) = t\, g(t)$. The frequency-differentiation property of the CTFT then yields $X(\omega) = i \frac{d}{d\omega} G(\omega)$.

From the CTFT table we know that

$$g(t) = \begin{cases} A & |t| \leq B \\ 0 & \text{e/w} \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(\omega) = \frac{2A \sin(\omega B)}{\omega}$$

so, with $A = 1/2$ and $B = 1$, we have $G(\omega) = \frac{\sin(\omega)}{\omega}$.

$$\frac{dG(\omega)}{d\omega} = \frac{\omega \cos \omega - \sin \omega}{\omega^2} \Rightarrow X(\omega) = i \, \frac{\omega \cos \omega - \sin \omega}{\omega^2}$$

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(ii) (5 Points) Evaluate $\int_{-\infty}^{+\infty} X(\omega)\, d\omega$. Show your work.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{i\omega t}\, d\omega \Rightarrow x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} X(\omega)\, d\omega = 2\pi\, x(0)$$

But $x(0) = 0 \Rightarrow \int_{-\infty}^{\infty} X(\omega)\, d\omega = 0$

(iii) (5 Points) Evaluate $\int_{-\infty}^{+\infty} |X(\omega)|^2\, d\omega$. Show your work.

According to Parseval's Identity (given in the CTFT table),

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-1}^{1} \frac{t^2}{4}\, dt = \frac{1}{2}\int_{0}^{1} t^2\, dt = \frac{t^3}{6}\Big|_0^1 = \frac{1}{6} \Rightarrow$$

$$\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \frac{\pi}{3}$$

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(b) (20 Points) Consider the continuous-time signal y described by

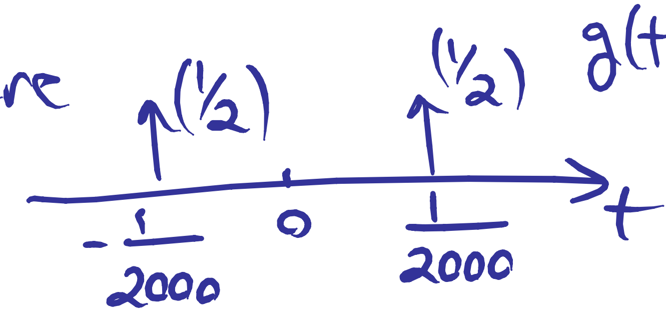
$$\forall t \in \mathbb{R}, \quad y(t) = \frac{1}{2} \frac{\sin\left[1000\pi\left(t + \frac{1}{2000}\right)\right]}{\pi\left(t + \frac{1}{2000}\right)} + \frac{1}{2} \frac{\sin\left[1000\pi\left(t - \frac{1}{2000}\right)\right]}{\pi\left(t - \frac{1}{2000}\right)}.$$

Determine a reasonably simple expression for, and provide a well-labeled plot of, $Y(\omega)$, the spectrum of the signal y.

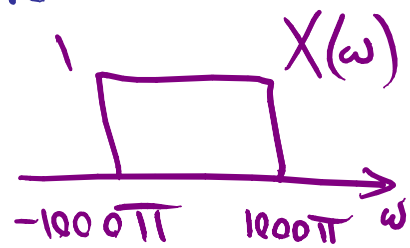
Let $x(t) = \frac{\sin(1000\pi t)}{\pi t}$. Then y can be thought of as follows: $y(t) = \frac{1}{2} x\left(t + \frac{1}{2000}\right) + \frac{1}{2} x\left(t - \frac{1}{2000}\right)$

$$= (x * g)(t)$$

where $g(t) = \frac{1}{2}\delta\left(t + \frac{1}{2000}\right) + \frac{1}{2}\delta\left(t - \frac{1}{2000}\right)$



Therefore, $Y(\omega) = X(\omega) G(\omega)$. From the CTFT tables we know that $x(t) = \frac{\sin(1000\pi t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$

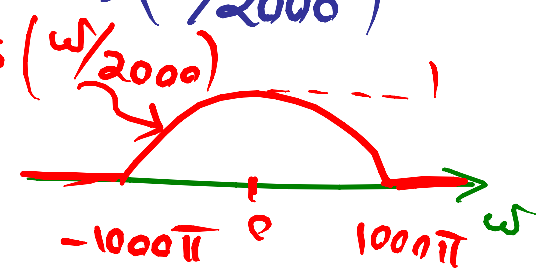


Moreover,

$$G(\omega) = \frac{1}{2} e^{i\omega/2000} + \frac{1}{2} e^{-i\omega/2000} = \cos(\omega/2000)$$

Clearly, then

$$Y(\omega) = \begin{cases} \cos\left(\frac{\omega}{2000}\right) & |\omega| \leq 1000\pi \\ 0 & \text{e/w} \end{cases}$$



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MT3.2 (20 Points) Consider an *aperiodic*, absolutely-integrable, continuous-time signal r whose spectrum $R(\omega)$ is well-defined and known. We construct another signal z as follows: $z(t) = \sum_{k=-\infty}^{+\infty} r(t - kT)$ for all $t \in \mathbb{R}$.

(a) (5 Points) Show that z is periodic and that its fundamental period p_z is no larger than T.

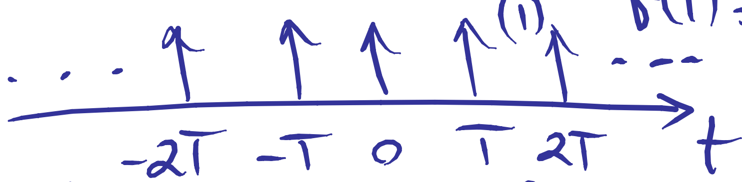
Let $\ell = k-1$

$$z(t+T) = \sum_{k=-\infty}^{\infty} r(t+T-kT) = \sum_{k=-\infty}^{\infty} r(t-(k-1)T) = \sum_{\ell=-\infty}^{\infty} r(t-\ell T) = z(t)$$

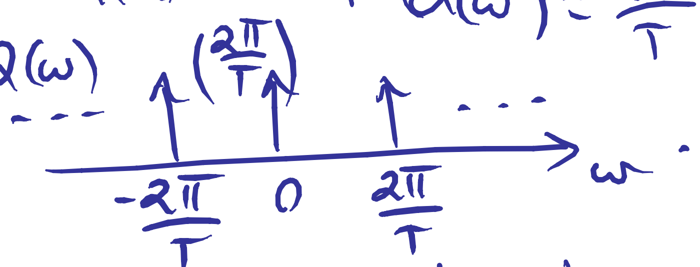
$\Rightarrow T$ is $\underline{\underline{a}}$ period of $z \Rightarrow$ Its fund. period cannot be larger.

(b) (15 Points) Let the continuous-time Fourier series (CTFS) expansion of z be $z(t) = \sum_{k=-\infty}^{+\infty} Z_k \, e^{ik\omega_0 t}$, where $\omega_0 = 2\pi/T$. Determine the CTFS coefficients Z_k. Interpret your result; what does your expression for Z_k mean?

$z(t) = \sum_{k=-\infty}^{\infty} r(t-kT) = (r * q)(t)$, where q is the impulse train $q(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$



We know that $Q(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$



In the tranform domain, we have $Z(\omega) = R(\omega) Q(\omega) \Rightarrow$

$$Z(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} R\left(\frac{2\pi k}{T}\right) \delta\left(\omega - \frac{2\pi k}{T}\right)$$

But $z(t) = \sum_{k=-\infty}^{\infty} Z_k e^{ik\omega_0 t} \Rightarrow Z(\omega) = 2\pi \sum_{k=-\infty}^{\infty} Z_k \delta\left(\omega - \frac{2\pi k}{T}\right)$

$$\Rightarrow \boxed{Z_k = \frac{1}{T} R\left(\frac{2\pi k}{T}\right)}$$

The Z_k's are de facto samples of the spectrum $R(\omega)$.

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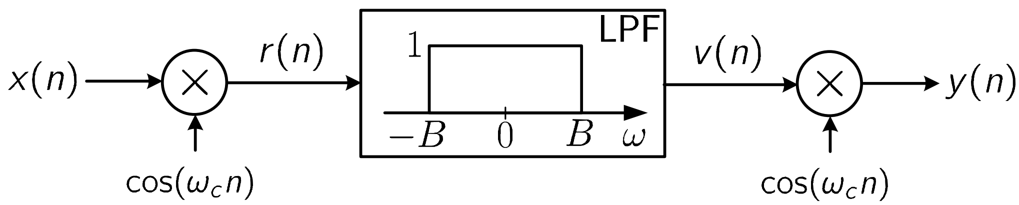
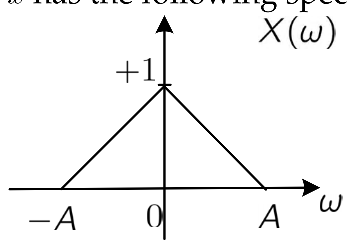
MT3.3 (40 Points) The figure below illustrates a discrete-time (DT) amplitude modulation and demodulation system in which the carrier frequency is ω_c and the cutoff frequency of the ideal DT low-pass filter is B.

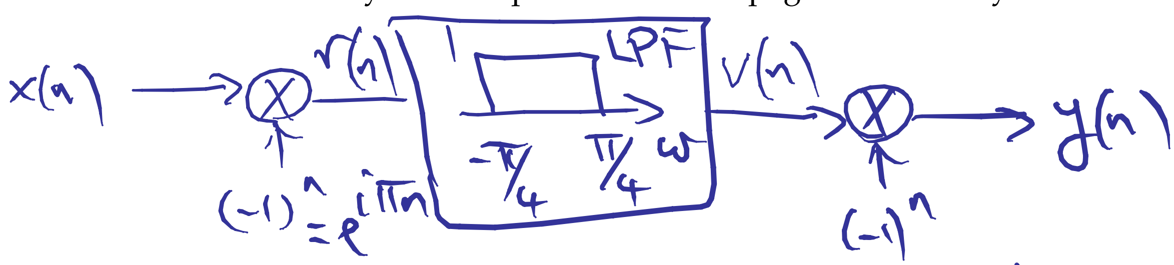


Suppose the DT input signal x has the following spectrum:



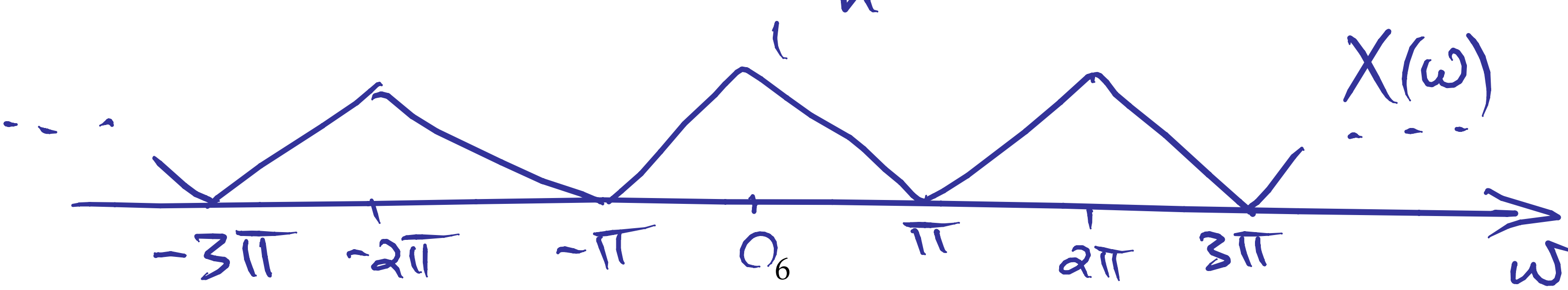
Although not shown in the diagram, it's understood that $X(\omega)$ is 2π-periodic. Also, needless to say, $0 < A \leq \pi$, $0 < \omega_c \leq \pi$, and $B > 0$.

(a) (20 Points) *For this part only*, let $A = \pi$, $\omega_c = \pi$, and $B = \pi/4$. Provide a well-labeled plot for each of $X(\omega)$, $R(\omega)$, $V(\omega)$, and $Y(\omega)$—the spectral values, respectively, of the signals x, r, v, and y—over the frequency interval $[-2\pi, 2\pi]$. You may use the space on the next page to continue your work for this part.



$$r(n) = (-1)^n x(n) \Rightarrow R(\omega) = \sum_n x(n)(-1)^n e^{-i\omega n}$$

$$R(\omega) = \sum_n x(n) e^{i\pi n} e^{-i\omega n} = \sum_n x(n) e^{-i(\omega-\pi)n} = X(\omega-\pi)$$



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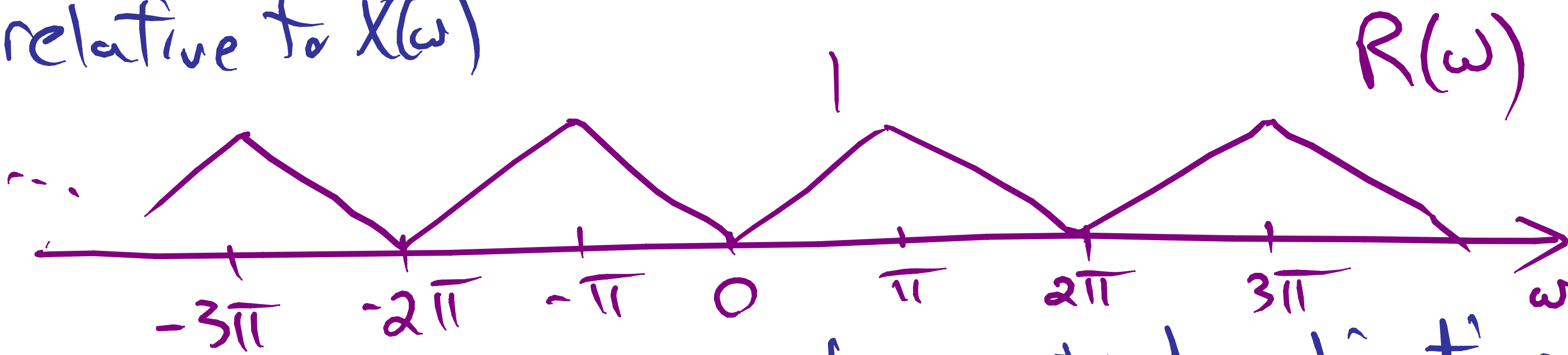
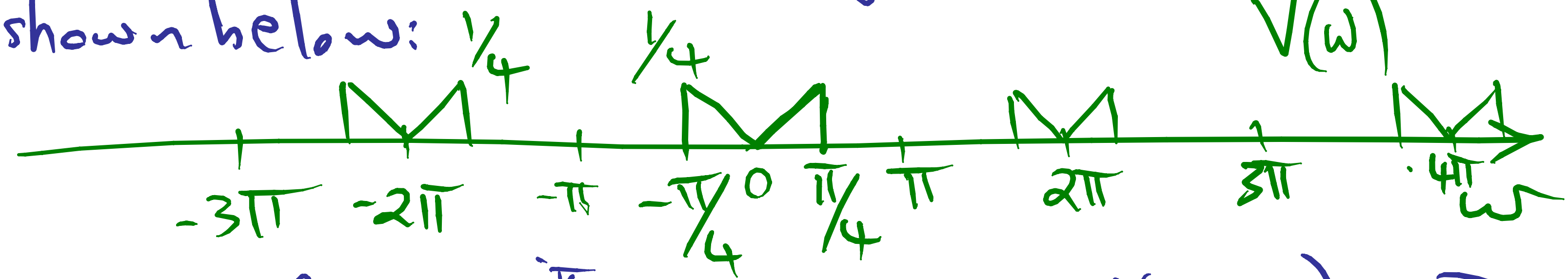
MT3.3(a) solutions continued on this page:

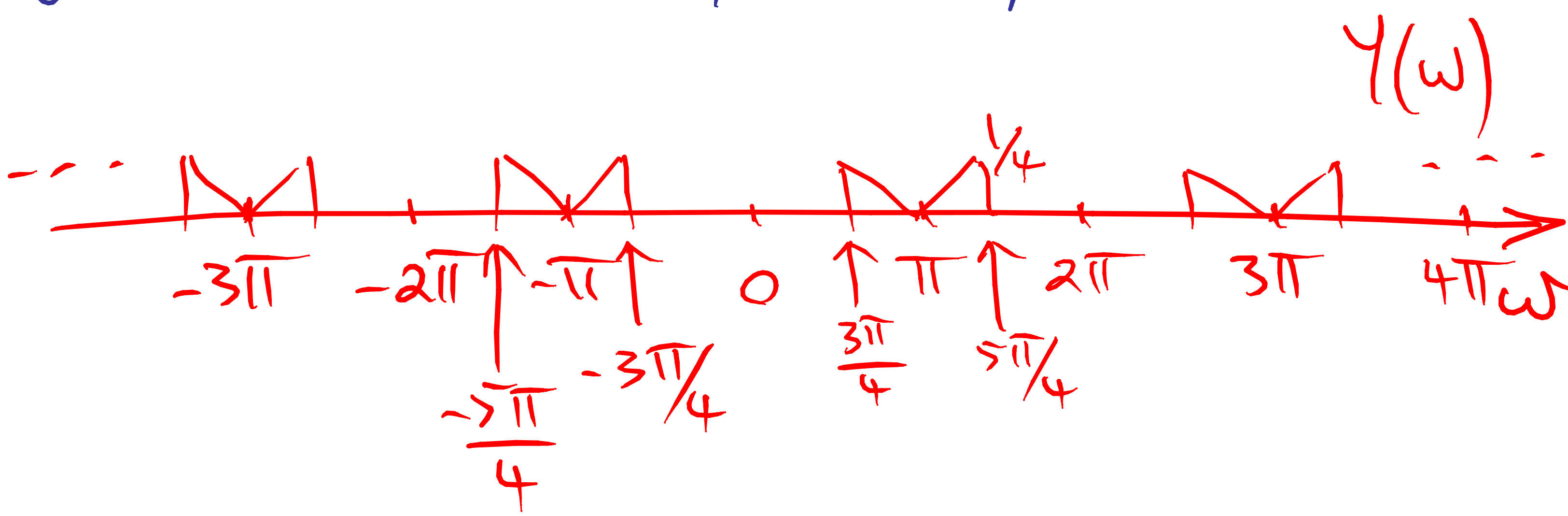
$R(\omega) = X(\omega - \pi)$, so this spectrum is shifted by π, relative to $X(\omega)$



The LPF truncates each spectral replication to within $\pm \pi/4$ of an integer multiple of 2π, as shown below:



$y(n) = (-1)^n v(n) = e^{i\pi n} v(n) \Longrightarrow Y(\omega) = V(\omega - \pi) \Longrightarrow$



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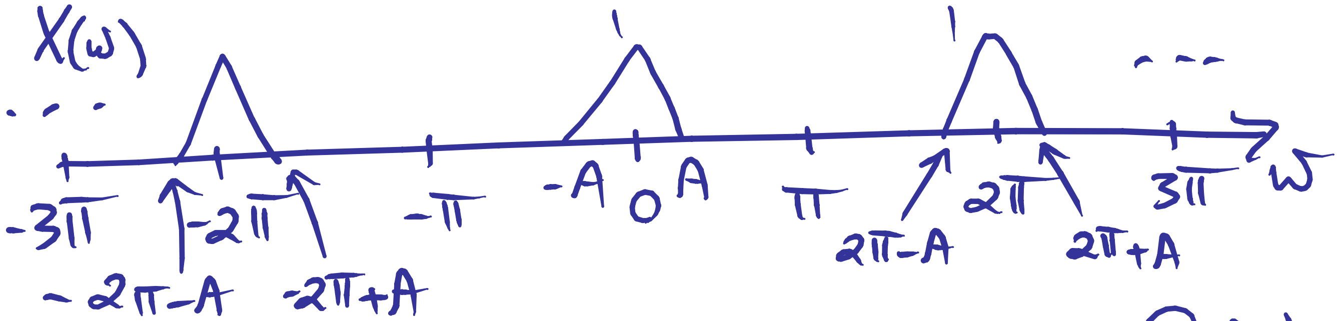
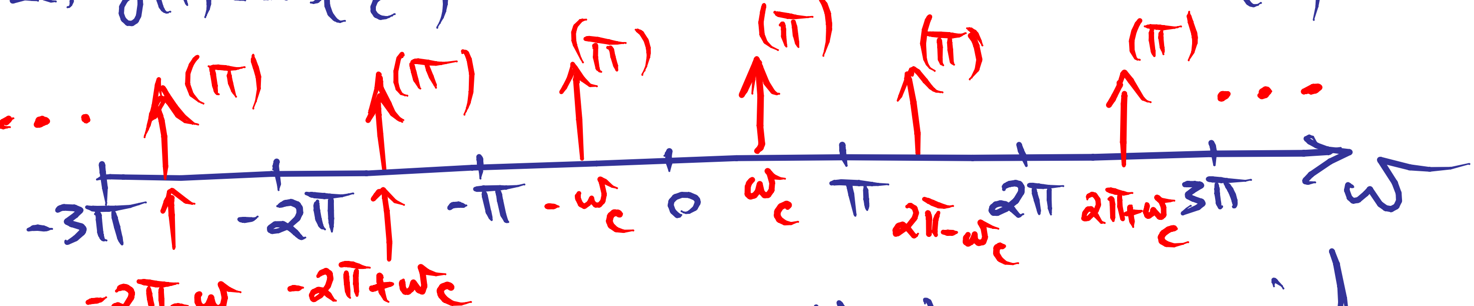
(b) (20 Points) Suppose $0 < A < \pi$. Determine the set of values of the carrier frequency ω_c such that any adjacent spectral replicas in $R(\omega)$ are guaranteed to *not* overlap.

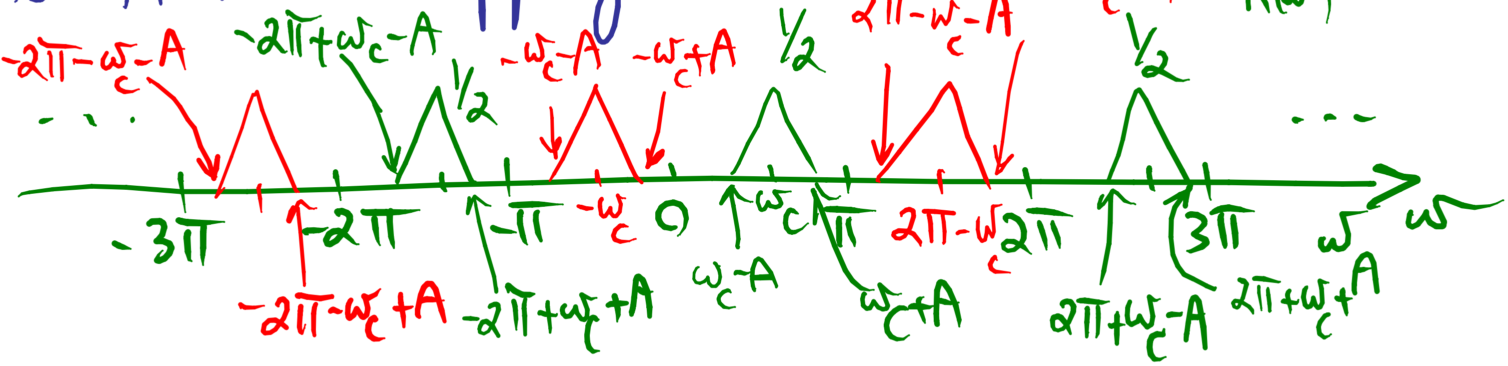


Let $g(n) = \cos(\omega_c n) \Rightarrow$ $G(\omega)$



$R(\omega) = \frac{1}{2\pi}(X \circledast G)(\omega) \Rightarrow$ We keep one period of $G(\omega)$ & convolve it w/ $X(\omega)$. This means

$R(\omega) = \pi[X(\omega+\omega_c) + X(\omega-\omega_c)]$, shown below for the non-overlapping case



To ensure no overlap, we must have $-\omega_c + A < \omega_c - A$

$\Rightarrow \omega_c > A$ AND $2\pi - \omega_c - A > \omega_c + A \Rightarrow \omega_c < \pi - A$

$\Longrightarrow$ $\boxed{A < \omega_c < \pi - A}$