Question 1. (15 points) c

Circle the correct answers (1 point each). Mark "T" only if the statement *must* be true without further assumptions. Justification is not needed, but incorrect answers carry a 1-point penalty, so random guessing does not help. You may leave any answer blank (0 points). You will also not get a negative total score on any group of five questions.

T F If the coefficient matrix A has a pivot position in every row, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. every column

T F The range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the column space of A. ✓

T F If the columns of an $n \times n$ matrix span $\mathbf{R}^n$, then they are linearly independent.
n vectors span $\mathbb{R}^n$ are indep.

T F For any $n \times n$ matrices A, B, C, we have $(ABC)^T = C^T B^T A^T$.
$C^T (AB)^T = C^T (B^T A^T)$

T F If a set of vectors in $\mathbf{R}^n$ is linearly dependent, then it contains more than n vectors. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

T F If the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the coefficient matrix A does not have a pivot position in every column. every row

T F The linear system $A\mathbf{x} = \mathbf{0}$ has more than one solution whenever there are free variables.

T F If S is a linearly dependent collection of vectors, then each vector in S is a linear combination of the other vectors.

T F The nullspace of a 3×4 matrix must contain infinitely many vectors.

T F If A is a 4×4 matrix and the system $A\mathbf{x} = \mathbf{e}_j$ is consistent for each vector of the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$ of $\mathbf{R}^4$, then A is invertible.

T F The dimensions of the row space and the column space of any $m \times n$ matrix A agree.

T F If one row in the echelon form of the augmented matrix of a system is $[0\ 0\ 0\ 1\ 0]$, then the system is inconsistent.

T F If a finite set S of vectors spans $\mathbf{R}^n$, then some subset of S forms a basis of $\mathbf{R}^n$.

T F The map $T : \mathbf{R}^2 \to \mathbf{R}^2$ which reflects points about the line $y = 1$ is linear.

T F The first row of a matrix product AB is the first row of A multiplied on the right by B.

Question 2. (12 points, 10+2)
(a) For the matrix A below, find bases of the nullspace, column space, row space and left nullspace. Make sure your method is clear.
(b) For what values of a, b does $[3, 5, a, b]$ lie in the row space? Explain.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 3 & 2 \\ 4 & 7 & 7 & 10 \end{bmatrix}$$

$\underbrace{\uparrow\ \uparrow}_{\text{pivot columns}}$

a)

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 4 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 2 & 0 & 1 & 0 & 0 \\ 4 & 7 & 7 & 10 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -6 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -6 & -4 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1+2R_2,\ R_2\cdot(-1)]{R_3-R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -8 & -3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 6 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$\underbrace{\uparrow\ \uparrow}_{\text{pivot columns}}$

$$\boxed{\text{basis of } Col(A) = \text{pivot columns of } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}}$$

$$\boxed{\text{basis of } Row(A) = \text{pivot rows in } rref(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 6 \end{bmatrix} \right\}}$$

$$3\begin{bmatrix} 1 \\ 0 \\ 0 \\ -8 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 6 \end{bmatrix} \checkmark$$

$$\boxed{\text{basis of } LNul(A) = \text{rows corresponding to 0 rows in } rref(A) = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

basis of Nul(A) is expressed in terms of free variables:

$$\begin{bmatrix} 1 & 0 & 0 & -8 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 - 8x_4 = 0 \\ x_2 + x_3 + 6x_4 = 0 \\ x_3 = x_3 \\ x_4 = x_4 \end{array} \longrightarrow \begin{array}{l} x_1 = 8x_4 \\ x_2 = -x_3 - 6x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array} \quad \vec{X} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{basis of } Nul(A) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

b)

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 1 & a \\ -8 & 6 & b \end{array}\right] \xrightarrow{R_4+8R_1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 1 & a \\ 0 & 6 & b+24 \end{array}\right] \xrightarrow[R_4-6R_2]{R_3-R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & a-5 \\ 0 & 0 & b-6 \end{array}\right]$$

to be in Row(A), this must be consistent, so $a-5=0$ and $b-6=$

$$\boxed{[3,5,a,b] \text{ is in } Row(A) \text{ for: } \quad a=5 \quad b=6}$$

Question 3. (10 points)
A linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ sends $[1,2]^T$ to $[1,2,3]^T$ and $[3,4]^T$ to $[3,4,5]^T$.
(a) Find $T([5,6]^T)$.
(b) Find a vector $\mathbf{v} \in \mathbf{R}^2$ with $T(\mathbf{v}) = [7,8,7]^T$, or else explain why no such vector exists.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right]$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2\,T\begin{pmatrix} 1 \\ 2 \end{pmatrix} + T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = -2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{2}\,T\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{2}T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$[T : \mathbb{R}^2 \to \mathbb{R}^3] \to [\vec{x} \longrightarrow A\vec{x}] \qquad A = \begin{bmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

a) $$T\begin{pmatrix} 5 \\ 6 \end{pmatrix} = A \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}}$$

b) $$T(\vec{v}) = \begin{bmatrix} 7 \\ 8 \\ 7 \end{bmatrix}$$

$$A \cdot \vec{v} = \begin{bmatrix} 7 \\ 8 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 8 \\ -1 & 2 & 7 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 2 & 14 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & -2 \end{array}\right]$$

No such vector $\vec{v}$ exists because the system $A\vec{v} = \begin{bmatrix} 7 \\ 8 \\ 7 \end{bmatrix}$ has a row $[0\ 0 \mid -2]$ which is inconsistent.

Question 4. (12 points)
Determine the inverse of the following matrix by one of two methods: either by row-reduction, or by using determinants. Check your answer.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 2 & 2 & 1 \end{bmatrix}$$

$$[A \mid I_3] \sim [I_3 \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 4 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -8 & -4 & 1 & 0 \\ 0 & -2 & -5 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \cdot (-1),\ R_3 \cdot (-1)]{\text{swap } R_2 \text{ and } R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 2 & 0 & -1 \\ 0 & 3 & 8 & 4 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{3}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 2 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & 1 & -1 & \frac{3}{2} \end{array}\right] \xrightarrow{R_3 \cdot 2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 & -2 & 3 \end{array}\right] \xrightarrow[R_2 - 5R_3]{R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 6 & -9 \\ 0 & 2 & 0 & -8 & 10 & -16 \\ 0 & 0 & 1 & 2 & -2 & 3 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_2]{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 7 \\ 0 & 1 & 0 & -4 & 5 & -8 \\ 0 & 0 & 1 & 2 & -2 & 3 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 3 & -4 & 7 \\ -4 & 5 & -8 \\ 2 & -2 & 3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 & 7 \\ -4 & 5 & -8 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 3-8+6 & -4+10-6 & 7-16+9 \\ 12-20+8 & -16+25-8 & 28-40+12 \\ 6-8+2 & -8+10-2 & 14-16+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$