Math 1B, Final Examination
N.Reshetikhin, May 15, 2015

Problem	1	2	3	4	5	6	7	8	9	10	11	12	*Total*
Points	15	15	15	15	15	15	15	15	15	15	15	15	180
Grade													

Student's Name:

GSI's name:

Student's i.d. number:

1.*15 pnts* Evaluate the integral

$$\int \sin(2x)e^{\sin x}dx = 2\int \sin x \cos x \, e^{\sin x} dx =$$

$$= \langle s = \sin x \rangle = \int s e^{s} ds = \int s \, d(e^{s}) =$$

$$= se^{s} - \int e^{s} ds = se^{s} - e^{s} + C =$$

$$= \sin x \, e^{\sin x} - e^{\sin x} + C$$

2.*15 pnts* Compute the integral

$$\int \frac{4dx}{(x-1)(x^2-1)}$$

$$x^2-1 = (x-1)(x+1)$$

$$\frac{1}{(x-1)(x^2-1)} = \frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{Ax+A + Bx^2-B + Cx^2-2xC+C}{(x-1)(x^2-1)}$$

$$\begin{array}{l|l} 1: 1 = A - B + C & \to 1 = 2C + C + C, \quad C = \frac{1}{4} \\ x: \ 0 = A - 2C & \left.\begin{array}{l} A = 2C \\ B = -C \end{array}\right\} \uparrow \quad A = \frac{1}{2} \\ x^2: 0 = B + C & \quad B = -\frac{1}{4} \end{array}$$

$$\int \frac{4dx}{(x-1)(x^2-1)} = \int \frac{2\,dx}{(x-1)^2} - \int \frac{dx}{x-1} + \int \frac{dx}{x+1} =$$

$$= -2\frac{1}{x-1} + \ln\left(\frac{x+1}{x-1}\right) + C$$

3. *15 pnts* Say whether each improper integral is convergent or divergent. Do not show your work. Each correct answer is worth *3 pnts* and each wrong answer is worth 0 pnts.

1. $\int_1^\infty \frac{dx}{x\ln x}$. $= \int_0^\infty \frac{d(\ln x)}{\ln x}$ divergent

2. $\int_1^\infty \frac{1}{x(\ln x)^2} dx$. $= \int_0^\infty \frac{d(\ln x)}{(\ln x)^2}$ divergent (at $x=1$)

3. $\int_0^\infty \frac{1-\cos x}{x^2} dx$. convergent

4. $\int_0^\infty \frac{\sin(x^2)}{x^2} dx$. convergent

5. $\int_0^1 \frac{dx}{x\sqrt{1-x}}$. divergent (at $x=0$)

4. *15 pnts* Find the radius and the interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{n+5}}(x-2)^n$$

$a_n = (-1)^n \frac{1}{\sqrt{n+5}}(x-2)^n$,

The ~~limit~~ ratio test:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n+5}}{\sqrt{n+6}}|x-2| = |x-2| \lim_{n\to\infty}\sqrt{\frac{n+5}{n+6}} = |x-2|$

$\Rightarrow \begin{cases} \text{series abs. conv. when } |x-2|<1 \\ \text{divergent } |x-2|>1 \end{cases} \Rightarrow \boxed{R=1}$

Boundary points:

1) $x-2=1$, $x=3$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+5}}$ conv. (conditionally), alt. series test

2) $x-2=-1$, $x=1$, $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+5}}$ divergent, integral test

$$\boxed{I=(1,3]}$$

5. *15 pnts* State whether each of the following series is absolutely convergent, conditionally convergent, or divergent. You do not have to show your work. Each correct answer is worth *3 pnts* and each wrong answer is worth 0 pnts.

1. $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$.

$\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, $\sum_{n\geq 1}^{\infty}(-1)^n\frac{1}{(\sqrt{n+1}+\sqrt{n})^2}$

conv. conditionally

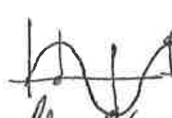
2. $\sum_{n=1}^{\infty}\cos(\frac{\pi}{2}+\frac{1}{n^2})$.

$\cos(\frac{\pi}{2}+\frac{1}{n^2}) = -\sin(\frac{1}{n^2})$, $\sum_{n\geq 1}^{\infty}\sin(\frac{1}{n^2})$ converges absolutely

3. $\sum_{n=1}^{\infty}\frac{n\sin n}{n^3+1}$.

$\left|\frac{n\sin n}{n^3+1}\right| \leq \frac{n}{n^3+1} \leq \frac{1}{n^2}$ conv. abs. (comp. test)

4. $\sum_{n=1}^{\infty}\tan(\frac{1}{n})\sin(\frac{\pi}{2}+\pi n)$.

$\sin(\frac{\pi}{2}+\pi n) =$  $=(-1)^{n+1}$

conv. conditionally (alt. series)

5. $\sum_{n=1}^{\infty}n^2\cos(\frac{\pi n}{2})$.

$\cos(\frac{\pi n}{2}) = \begin{cases} 0, & n\text{-odd} \\ (-1)^{\frac{n}{2}}, & n\text{-even} \end{cases}$ diverges

(divergence theorem)

6. *15 pnts* These are True-False questions. If the answer is True, you should explain why (concisely). If the answer is False, you should give a counter-example. Each problem is worth 5 points if the answer is correct and 0 points if the answer is not correct.

1. If the series $\sum_{n=1}^{\infty} a_n$ converges and the series $\sum_{n=1}^{\infty} a_n^2$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges conditionally.

   T

2. If the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges.

   F $\quad a_n = \frac{(-1)^n}{n}, \quad a_n+|a_n| = \begin{cases} 0, & n\text{-odd} \\ \frac{2}{n}, & n\text{-even} \end{cases} \Rightarrow \sum_n (a_n+|a_n|)$ diverges in this counterexample.

3. If the series $\sum_{n=1}^{\infty} a_n$ converges and the sequence $\{b_n\}_{n=1}^{\infty}$ converges as $n \to \infty$, and $b_n \neq 0$, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

   F $\quad a_n = \frac{(-1)^n}{\sqrt{n}}, \quad b_n = \frac{(-1)^n}{\sqrt{n}}$

- Assume opposite: $\sum_{n\geq 1}^{\infty} a_n$ converges absolutely. This would imply $\sum_{n\geq 1}^{\infty} a_n^2$ converges (by comparison test with $\sum_{n\geq 1}^{\infty} |a_n|$, but this would contradict $\sum_{n\geq 1}^{\infty} a_n^2$ diverges $\Rightarrow$ this is impossible
- We have only one alternative: $\sum_{n\geq 1}^{\infty} a_n$ converges conditionally.

7. *15 pnts* For each statement indicate whether it is true or false. You do not have to show your work. Each correct answer is worth *3 pnts* and each wrong answer is worth 0 pnts.

1. If a series $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges and R is the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-1)^n$, then $R \le 2$.

   $-1 \quad 0 \quad 1 \quad 2 \quad 3$

   diverges   center

   T

2. If $\sum_{n=1}^{\infty} c_n(x-1)^n$ converges at $x=3$, then $\sum_{n=1}^{\infty} c_n$ converges.

   $x-1=1$

   $1 \quad 3$

   center   converges

   T

3. The radius of convergence of $\sum_{n=1}^{\infty}(1+5^n)x^n$ is greater than 4.

   $R=\frac{1}{5}$ F

4. Even though the series $\sum_{n=1}^{\infty} c_n(x-1)^n$ converges at $x=-1$, the series $\sum_{n=1}^{\infty} c_n 2^n$ may diverge.

   can be I $=[-1,3)$

   $-1 \quad 0 \quad 1 \quad\quad 3$

   converges   center $x-1=2$

   T

5. If the series $\sum_{n=1}^{\infty} c_n x^n$ converges absolutely for $|x| \le 2$ then the radius of convergence is 2.

   F   it should diverge for $|x|>2$

8. *15 pnts* Find the general solution to the differential equation

$$yy' - y^2x = x\ .$$

$$yy' = y^2x + x = (y^2+1)x$$

separable

$$\frac{y\,dy}{y^2+1} = x\,dx\ ,$$

$$\int \frac{y\,dy}{y^2+1} = \int x\,dx\ , \quad \ln(y^2+1) = x^2 + C$$

$$y^2+1 = Ae^{x^2}\ , \quad \boxed{y^2 = Ae^{x^2} - 1}$$

$$\boxed{y = \pm\sqrt{Ae^{x^2} - 1}}$$

9. *15 pnts* Find the solution to the initial value problem

$$y'' + y = x^2 + e^x, \quad y(0) = -\frac{3}{2}, \quad y'(0) = \frac{1}{2}.$$

- $y'' + y = 0, \quad r^2 + 1 = 0, \quad r = \pm i,$

  $y_1 = \cos x, \quad y_2 = \sin x$

- $y'' + y = x^2$, particular solution

  $y_{p_1} = Ax^2 + Bx + C, \quad y_p'' = 2A$

  $2A + Ax^2 + Bx + C = x^2, \quad x^2: A = 1$

  $x: B = 0$

  $1: 2A + C = 0, \quad C = -2$

  $\boxed{y_{p_1} = x^2 - 2}$

- $y'' + y = e^x, \quad y_{p_2} = Ae^x, \quad y_{p_2}'' = Ae^x$

  $2Ae^x = e^x, \quad 2A = 1, \quad A = \frac{1}{2}$

  $\boxed{y_{p_2} = \frac{1}{2}e^x}$

- The general solution:

  $$y = C_1 \cos x + C_2 \sin x + x^2 - 2 + \frac{1}{2}e^x,$$

$y(0) = C_1 - 2 + \frac{1}{2} = C_1 - \frac{3}{2} = -\frac{3}{2}, \Rightarrow C_1 = 0$

$y'(0) = C_2 + \frac{1}{2} = \frac{1}{2}, \Rightarrow C_2 = 0$

$\boxed{y = x^2 - 2 + \frac{1}{2}e^x}$

9.*15 pnts* Find the solution to the initial value problem

$$y'' + y = x^2 + e^x, \quad y(0) = -\frac{3}{2}, \quad y'(0) = \frac{1}{2}.$$

- $y''+y=0$, $r^2+1=0$, $r=\pm i$,

  $y_1 = \cos x$, $y_2 = \sin x$

- $y''+y=x^2$, particular solution

  $y_{p_1} = Ax^2+Bx+C$, $y_p'' = 2A$

  $2A + Ax^2+Bx+C = x^2$, x^2: $A=1$

  x: $B=0$

  1: $2A+C=0$, $C=-2$

  $\boxed{y_{p_1} = x^2-2}$

- $y''+y = e^x$, $y_{p_2} = Ae^x$, $y_{p_2}'' = Ae^x$

  $2Ae^x = e^x$, $2A=1$, $A=\frac{1}{2}$

  $\boxed{y_{p_2} = \frac{1}{2}e^x}$

- The general solution:

$$y = C_1\cos x + C_2 \sin x + x^2 - 2 + \frac{1}{2}e^x,$$

$y(0) = C_1 - 2 + \frac{1}{2} = C_1 - 3/2 = -3/2$, $\Rightarrow C_1 = 0$

$y'(0) = C_2 + \frac{1}{2} = \frac{1}{2}$, $\Rightarrow C_2 = 0$

$$\boxed{y = x^2 - 2 + \frac{1}{2}e^x}$$

11. *15 pnts* Match pictures to differential equations.

a. $\dfrac{dy}{dx} = y^3 - x^3$

b. $\dfrac{dy}{dx} = \dfrac{y^2}{x^2}$

c. $\dfrac{dy}{dx} = -x + y$

d. $\dfrac{dy}{dx} = x^2 + y^2$

e. $\dfrac{dy}{dx} = y^2 - x^2$

12. *15 pnts* Find the power series solution to the initial value problem:

$$xy'' + xy = 0, \quad y(0) = 1, \quad y'(0) = 0 \; .$$

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

$$y = C_1 \cos x + C_2 \sin x, \quad y(0) = C_1 = 1$$

$$y'(0) = C_2 = 0$$

$$y = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$