Math W53 Final Exam solutions

1. Find the minimum and maximum values of the function

$$f(x, y, z) = x + y + z$$

subject to the constraint

$$x^2 + y^2 + 2z^2 \leq 10.$$

(You do not need to prove that minimum and maximum values exist.)

Solution. Candidates for minima and maxima are critical points of f on the interior of the region $x^2+y^2+z^2 \leq 10$, and minima and maxima on the boundary. Since $f_x = f_y = f_z = 1$, which never vanishes, the function f has no critical points. The boundary of the region is the surface $g(x, y, z) = x^2 + y^2 + 2z^2 = 10$. We find minima and maxima on this surface using Lagrange multipliers. The Lagrange multiplier equation $\nabla f = \lambda \nabla g$ gives $1 = 2\lambda x = 2\lambda y = 4\lambda z$. It follows that $x = y = 1/(2\lambda)$ and $z = 1/(4\lambda)$. Plugging in to the constraint equation $g(x, y, z) = 10$ gives $5/(8\lambda^2) = 10$, so $\lambda = \pm 1/4$. We then obtain the two points $(x, y, z) = (2, 2, 1)$ and $(x, y, z) = (-2, -2, -1)$. On the first point $f = 5$, and on the second point $f = -5$. Thus the minimum is -5 and the maximum is 5.

2. Let S be a surface which is contained in the plane $z = x + y$, oriented upward. Suppose that S has area 2018. Consider the constant vector field $\mathbf{F} = \langle 3, 4, 5\rangle$. Calculate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Solution. By the definition of flux, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS$, where $\mathbf{n}$ is the oriented unit normal vector. Since the surface is the graph of $g(x, y) = x+y$, an oriented normal vector is $\langle -g_x, -g_y, 1\rangle = \langle -1, -1, 1\rangle$. Normalizing this we obtain $\mathbf{n} = \frac{1}{\sqrt{3}}\langle -1, -1, 1\rangle$. Then $\mathbf{F} \cdot \mathbf{n} = -2/\sqrt{3}$. Thus $\iint_S (\mathbf{F} \cdot \mathbf{n}) dS = \iint_S (-2/\sqrt{3}) dS = (-2/\sqrt{3})\,\text{area}(S) = -4036/\sqrt{3}$.

3. Calculate the iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dy\, dx.$$

Solution. We change the order of integration. The integral is over the region $0 \le x \le 4$ and $\sqrt{x} \le y \le 2$. Thus it is bounded by the y axis, the line $y = 2$, and the curve $y = \sqrt{x}$. When $y \ge 0$, the latter equation is equivalent to $x = y^2$. Thus the integral is

$$\int_0^2 \int_0^{y^2} e^{y^3}\,dx\,dy = \int_0^2 y^2 e^{y^3}\,dy = \frac{1}{3}e^{y^3}\Big|_{y=0}^{y=2} = \frac{1}{3}(e^8 - 1).$$

4. Let E be the region defined by the inequalities

$$x^2 + y^2 + z^2 \le 4, \quad 0 \le y \le x, \quad z \ge 0.$$

Calculate the total mass of E if the mass density is z^2.

Solution. In spherical coordinates, the region E is defined by the inequalities $0 \le \rho \le 2$, $0 \le \theta \le \pi/4$, and $0 \le \phi \le \pi/2$. Thus the total mass is

$$\begin{aligned}
\int_0^{\pi/2} \int_0^{\pi/4} \int_0^2 (\rho\cos\phi)^2 \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/2} \int_0^{\pi/4} \int_0^2 \rho^4 \cos^2\phi \sin\phi\,d\rho\,d\theta\,d\phi \\
&= \frac{32}{5} \int_0^{\pi/2} \int_0^{\pi/4} \cos^2\phi \sin\phi\,d\theta\,d\phi \\
&= \frac{8\pi}{5} \int_0^{\pi/2} \cos^2\phi \sin\phi\,d\phi \\
&= \frac{-8\pi}{15} \cos^3\phi\Big|_{\phi=0}^{\phi=\pi/2} \\
&= \frac{8\pi}{15}.
\end{aligned}$$

5. Find the tangent plane to the parametrized surface

$$\mathbf{r}(u, v) = \langle u^2 - 1, uv, v^3 \rangle$$

at the point $(3, 4, 8)$. Please give an answer of the form $ax + by + cz = d$.

Solution. The equation for the plane is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ where $\mathbf{r}_0 = \langle 3, 4, 8 \rangle$ and $\mathbf{n}$ is a normal vector to the plane at this point. The point $\langle 3, 4, 8 \rangle$ corresponds to $u = v = 2$. A normal vector is then $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ evaluted at $(u, v) = (2, 2)$. We have

$$\begin{aligned}
\mathbf{r}_u(2, 2) &= \langle 2u, v, 0 \rangle|_{(u,v)=(2,2)} = \langle 4, 2, 0 \rangle, \\
\mathbf{r}_v(2, 2) &= \langle 0, u, 3v^2 \rangle|_{(u,v)=(2,2)} = \langle 0, 2, 12 \rangle.
\end{aligned}$$

Then $\mathbf{n} = \langle 4, 2, 0\rangle \times \langle 0, 2, 12\rangle = \langle 24, -48, 8\rangle$. To make the arithmetic simpler we can rescale this to $\mathbf{n} = \langle 3, -6, 1\rangle$. The equation for the plane is then

$$\langle 3, -6, 1\rangle \cdot \langle x - 3, y - 4, z - 8\rangle = 0$$

which expands out to

$$3x - 6y + z = -7.$$

(Any nonzero constant multiple multiple of this equation would also be correct.)

6. Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 2x + 3y$, oriented counterclockwise when viewed from above. Let

$$\mathbf{F} = \langle x^{2018} + y, y^{2018} + z, z^{2018} + x\rangle.$$

Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution. We use Stokes theorem. Let S be the part of the plane $z = 2x + 3y$ where $x^2 + y^2 \leq 1$, oriented upward. Then by Stokes theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S \langle -1, -1, -1\rangle \cdot d\mathbf{S}.$$

We can parametrize the surface S as $\mathbf{r}(x, y) = \langle x, y, 2x + 3y\rangle$. Then $\mathbf{r}_x \times \mathbf{r}_y = \langle -2, -3, 1\rangle$ so

$$\begin{aligned}\iint_S \langle -1, -1, -1\rangle \cdot d\mathbf{S} &= \iint_{x^2+y^2\leq 1} \langle -1, -1, -1\rangle \cdot \langle -2, -3, 1\rangle dA \\ &= \iint_{x^2+y^2\leq 1} 4 dA \\ &= 4\pi.\end{aligned}$$

7. Let D be the region in the plane where $x^2 + 2y^2 \leq 1$. Calculate the double integral

$$\iint_D (x^2 + 2y^2)^{2018} dA.$$

Solution. We make the change of variables $x = u, y = v/\sqrt{2}$. Then the region D in the x, y plane corresponds to the region $u^2 + v^2 \leq 1$ in the u, v plane. The Jacobian of the transformation is $1/\sqrt{2}$. Thus we get

$$\iint_D (x^2 + 2y^2)^{2018} DA = \frac{1}{\sqrt{2}} \iint_{u^2+v^2 \leq 1} (u^2 + v^2)^{2018} dA.$$

We evaluate the integral on the right using polar coordinates in the u, v plane to get

$$\frac{1}{\sqrt{2}} \int_0^{2\pi} \int_0^1 r^{4037} dr = \left(\frac{1}{\sqrt{2}}\right)(2\pi)\left(\frac{1}{4038}\right) = \frac{\pi}{2019\sqrt{2}}.$$

8. Calculate the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the hemisphere

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

oriented upwards, and

$$\mathbf{F} = \langle x + \sin y, y + \cos z, z + 1 \rangle.$$

Solution. Let E be the half-ball $x^2 + y^2 + z^2 \leq 1, z \geq 0$, and let S' be the surface $x^2 + y^2 \leq 1, z = 0$, oriented upward. Then by the Divergence Theorem, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\operatorname{div} \mathbf{F}) dV + \iint_{S'} \mathbf{F} \cdot d\mathbf{S}.$$

Now $\operatorname{div} \mathbf{F} = 3$, so the first integral on the right is 3 times the volume of the half-ball, which is 2π. And the second integral on the right is the integral of 1 over the unit disk, which is π. Thus the answer is 3π.

9. Suppose a parametrized curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $0 \leq t \leq 1$ satisfies the equation

$$xx'(t) + yy'(t) + zz'(t) = 0$$

for all t. If $x(0) = y(0) = z(0) = 3$ and $x(1) = y(1) = 2$, find $|z(1)|$.

Solution. The given equation (times 2) is equivalent to

$$\frac{d}{dt}(x^2 + y^2 + z^2) = 0.$$

It follows that $x^2 + y^2 + z^2$ is constant on the curve. Since this constant equals 27 at $t = 0$, it also equals 27 at $t = 1$. Since $x(1)^2 + y(1)^2 = 8$, we must have $z(1)^2 = 19$, so $|z(1)| = \sqrt{19}$.

10. Let f be a differentiable function on $\mathbb{R}^2$ such that

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y)$$

for all x, y. Suppose also that $f(2, 3) = 6$. Compute $f(4, 1)$. As usual, justify your answer.

Solution. Let C be the line segment from $(2, 3)$ to $(4, 1)$. For any parametrization of this curve, we have $x'(t) = -y'(t)$. It then follows that $\int_C (\nabla f) \cdot d\mathbf{r} = 0$. So by the Fundamental Theorem of Line Integrals, $f(4, 1) = f(2, 3) = 6$.

Alternate solution. By the given information and the Chain Rule, $\frac{d}{dt} f(t, 5 - t) = 0$, so $f(t, 5 - t)$ does not depend on t. Plugging in $t = 4$ and $t = 2$ gives $f(4, 1) = f(2, 3)$ as before.