1 36.8

2 27

3 30

93.8

1. The following equation applies to a plane wall of half thickness L:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Assume initial condition

$$T(x, 0) = T_i$$

and the boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

and

$$\frac{\partial^2 \theta^*}{\partial x^{*2} \cdot L^2} = \frac{1}{\alpha}\frac{\partial \theta^*}{\partial t}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L,t) - T_\infty]$$

a. Re-write the equation above using non-dimensional length (x/L) and non-dimensional temperature to find the non-dimensional time (the Fourier number).

3

$x^* = \frac{x}{L}$; $\theta^* = \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$; $Fo = \frac{\alpha t}{L^2} = t^*$

$\partial x^* = \frac{\partial x}{L}$ $\partial \theta^* = \frac{\partial T}{(T_i - T_\infty)}$ $\partial Fo = \frac{\alpha\, \partial t}{L^2}$

$\partial x = L\,\partial x^*$ $\partial T = \partial\theta^*(T_i - T_\infty)$ $\partial t = \frac{L^2}{\alpha}\partial Fo$

$$\frac{\partial^2 \theta^* (T_i - T_\infty)}{\partial x^{*} \, L^2} = \frac{1}{\alpha}\cdot\frac{\partial\theta^*(T_i - T_\infty)}{\frac{L^2}{\alpha}\,\partial Fo}$$

$$\boxed{\frac{\partial^2 \theta^*}{\partial x^*} = \frac{\partial \theta^*}{\partial Fo}}$$

b. Non-dimensionalize the initial and boundary conditions to derive the Biot number.

5

$$\boxed{\theta^*(x^*, 0) = 1}$$

$$\boxed{\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0}$$

$$-k\frac{\partial \theta^* (T_i - T_\infty)}{\partial x^* \; L} = h[T(L, T) - T_\infty]$$

$$\boxed{\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1, t^*)}$$

c. Provide a detailed explanation of the physical significance of the Bi and Fo numbers including an explanation of the significance of high vs. low values for the Bi and Fo numbers.

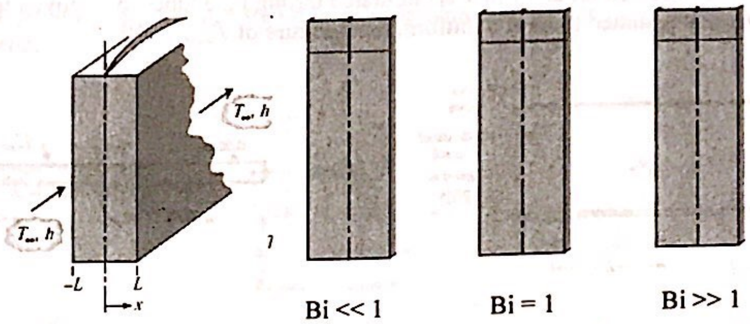
5

Bi ⇒ ratio of conduction thermal resistance over convection thermal resistance. small Biot number indicate temperature gradient on body by conduction is negligible compared to temperature gradient caused by convection (to determine if lumped or not). high Bi number means the opposite, temperature gradient caused by conduction is more significant compared to temperature gradient caused by convection

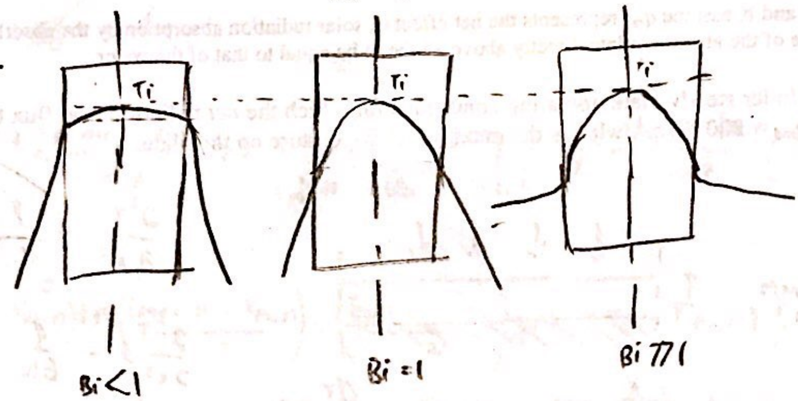
Fo ⇒ $\frac{kt}{\rho c (L_c)^2}$ ⇒ ratio of conduction over stored energy rate. small Fo number means not enough time to stored the thermal energy. high Fo means enough time for heat to be stored

Scanned with CamScanner

d. **Show the transient temperature distribution in a plane wall symmetrically cooled by convection as shown in the figures below for Bi << 1, Bi = 1, and Bi >> 1. The horizontal line indicates temperature at time t = 0 as $T_{initial}$.**



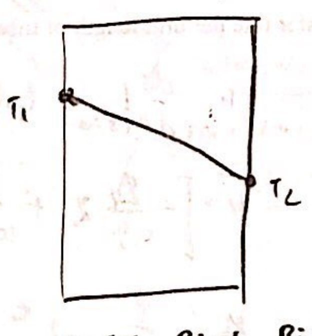
5



e. **Show the steady state temperature distribution in a plane wall with one surface at T1 and the other surface at T2 for Bi << 1, Bi = 1, and Bi >> 1.**

~~22~~ 5 $\frac{\partial^2 T}{\partial x^2} = 0 \rightarrow T = C_1 x + C_2$

assuming : • $T_1 > T_2$
• no $\dot{g}$
• Same thickness

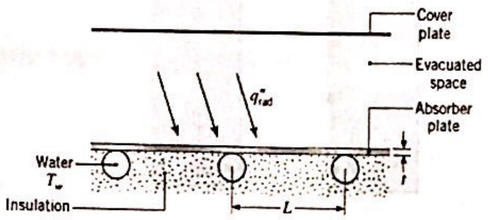


for Bi<<1, Bi =1, Bi >>1

Since $T(x) = C_1 x + C_2$
no dependence on K or h
Same temperature distribution at steady state

$.4\left(\frac{23}{25}\right) = .368$

2. Copper tubing is joined to the absorber of a flat-plate solar collector as shown. The aluminum alloy absorber plate is 6 mm thick and well insulated on its bottom (k = 180 W/mK). The top surface of the plate is separated from a transparent cover plate by an evacuated space. The tubes are spaced a distance L of 0.20 m from each other, and water is circulated through the tubes to remove the collected energy. The water may be assumed to be at a uniform temperature of T_{water} = 60 C.

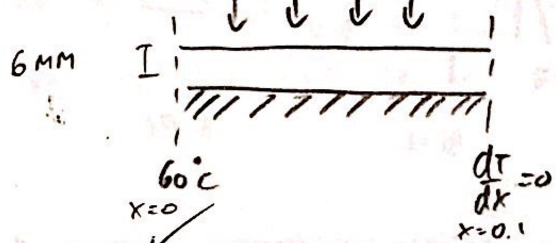


For parts a and b, assume q_{rad} represents the net effect of solar radiation absorption by the absorber plate and the temperature of the absorber plate directly above a tube to be equal to that of the water.

a. Under steady-state operating conditions for which the *net* radiation heat flux to the surface is q_{rad} = 800 W/m², what is the maximum temperature on the plate.

5

$q_{rad} = 800 \ W/m^2$



$\frac{\partial^2 T}{\partial x^2} + \frac{q \cdot A}{V \cdot k} = 0$

$\frac{\partial^2 T}{\partial x^2} = -\frac{q}{tk}$

$\frac{\partial T}{\partial x} = -\frac{q}{tk} x + C_1$

$T = -\frac{q}{2tk} x^2 + C_1 x + C_2$

BC: $\frac{\partial T}{\partial x}\Big|_{x=0.1} = 0$

$-\frac{q \cdot 0.1}{tk} + C_1 = 0$

$C_1 = \frac{q}{10tk}$

$T(0) = 60°C$

$T_2 = 60°C = 333K$

$T(x) = -\frac{q}{2tk} x^2 + \frac{q}{10tk} x + 333$

$\frac{\partial T}{\partial x}\Big|_{x=0.1} = 0 \rightarrow T_{max} \ (x = 0.1m)$

$T(x) = -\frac{800}{2(0.006)(180)} \times (0.1)^2 + \frac{800}{10(0.006)(180)} \times 0.1 + 333$

$\boxed{T_{max} = 339.667 \text{ kelvin} = 66.667 \ °C}$

b. What is the heat transfer rate per unit length of tube?

4

$q_{tube} \cdot L = -k \frac{dT}{dx}\Big|_{x=0} \ t \cdot L \ \times 2$

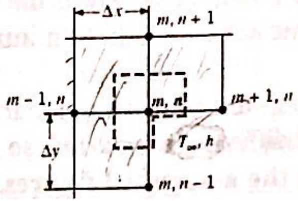
$q_{tube} = -k \left[-\frac{q}{tk} x + \frac{q}{10tk}\right]_{x=0} \cdot t$

$q'_{tube} = \frac{-k \left(\frac{\partial T}{\partial x}\right]_{x=0} \cdot t \cdot L}{L}$

$= -k\left(\frac{q}{10tk}\right) t = \boxed{-80 \ W/m}$ (Minus since cooling down the plate)

=

c. **Assume the portion of the tube making contact between the water tube and the plate can be represented by the finite difference surface shown in the figure below where points m-1, n and m, n+1 are aluminum with thermal conductivity k. Assume the temperature of the plate and the temperature of the water are no longer the same. Derive the corresponding finite difference equation.**



5

$$\dot{E}_{in} + \cancel{\dot{E}_g} = 0$$

$$k\frac{T_{m-1,n}-T_{m,n}}{\Delta x}(\Delta y\cdot 1) + k\frac{T_{m,n+1}-T_{m,n}}{\Delta y}(\Delta x\cdot 1) + k\frac{T_{m+1,n}-T_{m,n}}{\Delta x}\left(\frac{\Delta y}{2}\cdot 1\right) + k\frac{T_{m,n-1}-T_{m,n}}{\Delta y}\left(\frac{\Delta x}{2}\cdot 1\right) + h(T_\infty - T_{m,n})\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)$$

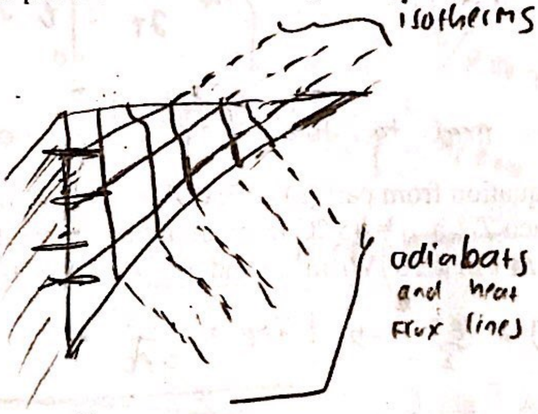
assumin $\Delta y = \Delta x$

$$k(T_{m-1,n} - T_{m,n}) + k(T_{m,n+1} - T_{m,n}) + k\left(\frac{T_{m+1,n} - T_{m,n}}{2}\right) + k\left(\frac{T_{m,n-1} - T_{m,n}}{2}\right) + h(T_\infty - T_{m,n})(\Delta x) = 0$$

$$T_{m,n} = \frac{k\left(T_{m-1,n} + T_{m,n+1} + \frac{T_{m,n-1}}{2} + \frac{T_{m+1,n}}{2}\right) + h(T_\infty)(\Delta x)}{3k + h\cdot\Delta x}$$

d. **Create a flux plot of this element showing isothermal lines, adiabats, and heat flux lines.**

4



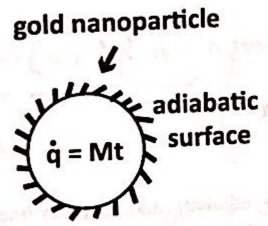
$$.3\left(\frac{18}{20}\right) = .27$$

3. *Gold nanoparticles* have numerous biological and medical applications. One common application is *photothermal therapy*, in which gold nanoparticles are deposited into cancerous tissue. Then, a laser whose energy is strongly absorbed by the gold nanoparticles, but not by human tissue, is used to raise the temperature of the gold nanoparticles and, consequently, the temperature of the surrounding cancer cells. The goal is to heat the nanoparticles no higher than the $T_{threshold}$ necessary to achieve cell death, to minimize damage to healthy tissue.

In this problem we will consider spherical gold nanoparticles deposited in a cancerous tumor. We will consider ***several different scenarios***, **so please read each part of this problem carefully and refer to the associated figures.**

a) Consider a single nanoparticle of diameter d, which you can assume to be thermally lumped. At time $t = 0$, the laser is turned on and the power ramps up linearly such that the gold nanoparticles experience uniform volumetric generation of $\dot{q} = Mt$, where M is a constant with units of [W/(m^3-s)] or, equivalently, [J/(m^3-s^2)]. The nanoparticle is initially at a temperature T_i. For the timescales on which the laser heating occurs, the nanoparticle surface can be approximated as adiabatic. Set up, but do not yet solve, the governing differential equation and the initial condition needed to solve for the nanoparticle temperature, T_{NP}, as a function of time t.

$$\rho \cancel{V} C \frac{\partial T}{\partial t} = \dot{q} \cancel{V}$$



$$\rho C \frac{\partial T}{\partial t} = Mt \quad ✓$$

$$\frac{\rho C}{M} \partial T = t\, \partial t \quad ✓$$

$$\frac{\rho C}{M} \int_{T_i}^{T_{NP}} \partial T = \int_0^t t\, \partial t \qquad 4/4$$

need to know T_{NP} or t (one or another)

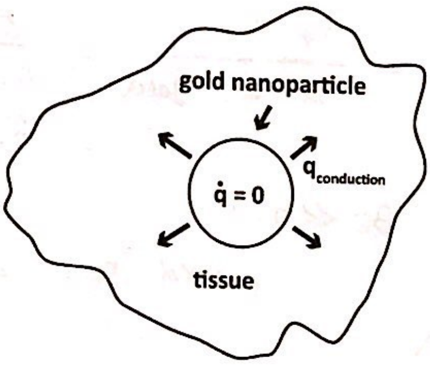
b) Now solve your equation from part (a) to find $T_{NP}(t)$. How long will it take the nanoparticle to reach $T_{threshold} = 45$ °C from an initial temperature of $T_i = 37$ °C? Take $d = 100$ nm, $M = 4 \times 10^9$ W/(m^3-s) and $\rho c_{gold} = 2.5 \times 10^6$ J/(m^3-K).

$$\frac{\rho C}{M}[T_{NP} - T_i] = \frac{t^2}{2} \rightarrow \boxed{T_{NP} = \frac{M t^2}{2\rho C} + T_i} \quad ✓$$

$$t = \sqrt{\frac{2\rho C}{M}[T_{NP} - T_i]} \rightarrow \boxed{t(45°C) = 0.1 \text{ Second}} \quad ✓ \qquad 3/3$$

c) At a later time, the laser is shut off and $\dot{q} = 0$. The nanoparticle (initially at $T_{NP} = T_{threshold}$) now cools by heat conduction into the surrounding tissue, which is at a temperature of $T_{tissue} = 37$ °C. The thermal resistance for the nanoparticle losing heat via conduction to the tissue can be approximated as $R_{NP} = 1/(2dk_{tissue})$. Take $k_{tissue} = 0.6$ W/(m-K). What is T_{NP} 5 nanoseconds (1 nanosecond = 10^{-9} s) after the laser is shut off? Once again, assume the nanoparticle to be thermally lumped.

Note: You may either set up and solve a new differential equation, or you may simply write down the answer by replacing the *convection resistance* in the solution derived in your textbook/in class with the appropriate *conduction resistance* for this problem. Recall that the thermal time constant $\tau_{thermal} = (1/hA)*(\rho Vc)$, where $(1/hA)$ is a thermal resistance and (ρVc) is a thermal capacitance.



$$-\frac{T - T_\infty}{R_{NP}} = \underbrace{\rho V c}_{C_t} \frac{\partial T}{\partial t}$$

$$\theta = T - T_{tissue}$$

$$-\frac{\theta}{R_{NP}} = C_t \frac{\partial \theta}{\partial t}$$ ✓

$T_i = 45°C$

$d = 100$ nm

$$\int_0^t -\frac{\partial t}{R_{NP} \cdot C_t} = \int_{\theta_i}^{\theta} \frac{\partial \theta}{\theta}$$

$$-\frac{t}{R_{NP} \cdot C_t} = \ln\left(\frac{\theta}{\theta_i}\right)$$

$$\exp(-t/R_{NP} \cdot C_t) = \frac{\theta}{\theta_i}$$

✓

$$\exp(-t/R_{NP} \cdot C_t) = \frac{T - T_{tissue}}{\underbrace{T_i - T_{tissue}}_{45-37}}$$

$$\boxed{T_{5ns} = 42.0585°C}$$

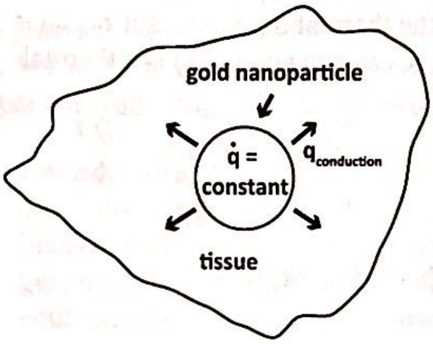
$$\exp\left(-\frac{2t\,dk}{\rho V c}\right)$$ ✓

$$0.632 = \frac{T - T_{tissue}}{8°C}$$

4/4

d) Consider a *different* scenario in which a constant laser power is used to maintain $T_{NP} = T_{threshold}$ (i.e. $\dot{q}$ is constant and no longer time-dependent). The nanoparticle loses heat via conduction to the tissue, as in part (c). Although we technically cannot apply the thermal resistor concept to the thermal resistance *within* the nanoparticle due to the volumetric energy generation, it turns out that in this case we can still define an "effective" thermal resistor for the internal thermal resistance of the nanoparticle, where $R_{internal} = 1/(2\pi d k_{gold})$.

Given $k_{gold} = 315$ W/(m-K), is it valid to assume the nanoparticle is thermally lumped in this scenario? *Hint:* Recall the most general definition of the Biot number.



$$Bi = \frac{R_{cond\ internal}}{R_{xip}}$$

$$= \frac{2d\, k_{tissue}}{2\pi d\, k_{gold}} = \frac{0.6}{\pi \cdot 315}$$

$$= 6.06 \times 10^{-4}$$

$Bi \ll 0.1$

lumped still valid

4/4

e) Now consider gold nanoparticles with a smaller diameter of $d = 10$ nm. Given that the primary heat carriers in gold have a heat capacity of 1.9×10^4 J/(m^3-K) and a velocity of 1.4×10^6 m/s, will the thermal conductivity of these smaller gold nanoparticles differ from the bulk value of $k_{gold} = 315$ W/(m-K)? If so, for the scenario given in part (d), can these smaller nanoparticles still be considered thermally lumped? Justify your answer with a calculation and a one-sentence explanation.

$$315 = \frac{1}{3} \times 1.9 \times 10^4 \times 1.4 \times 10^6 \times \Lambda$$

$$\Lambda_{mfp} = 35.52 \text{ nm}$$

3/3

Since $\Lambda_{mfp} > d$

$$k = \frac{1}{3} \times 1.9 \times 10^4 \times 1.4 \times 10^6 \times 10\text{ nm}$$

$$k = 88.667$$

$$Bi = \frac{0.6}{\pi \cdot 88.667} = 2.5 \times 10^{-3}$$

$Bi < 0.1$ still lumped

f) Given that the electrical conductivity of gold is approximately $\sigma = 4.1 \times 10^7\ \Omega^{-1}$-m^{-1} at 300 K, which energy carriers carry most of the heat in gold?

$$\frac{k_{el}}{\sigma T} = L_0$$

$$k_{el} = 2.44 \times 10^{-8} \frac{V^2}{K^2} \times 4.1 \times 10^7 \times 300 = 300.12$$

2/2

$$k_{el} + k_{phonon} = k_{gold}$$

300.12   14.88   315

$k_{el} \gg k_{phonon}$

electron carries more heat in gold

p3: 20/20 = .3

nicely done