Problem 1

(a) The inner surface near the $4Q$ charge has a total charge $-4Q$ distributed unevenly on its surface. The inner sufrace near the $-2Q$ charge has a total charge $2Q$ distributed unevenly on its surface. Finally, the outer surface has a charge $2Q$ distributed **evenly** on its surface.

(b) Outside the sphere, the electric field is that of a point charge located at the center of the conducting sphere with charge $q = 4Q - 2Q = 2Q$. If we consider an infinitesimal length of rod with charge dq located a distance x away from the center of the sphere, the force on the segment is given by

$$d\vec{F} = \left(\frac{1}{4\pi\epsilon_0}\frac{2Q}{x^2}dq\right)\hat{x} \tag{1}$$

Using $dq = \lambda dx$, where $\lambda = \frac{-3Q}{L}$, we find that the total force on the rod is

$$\vec{F} = \hat{x}\left(\frac{2Q\lambda}{4\pi\epsilon_0}\right)\int_{3R}^{3R+L}\frac{dx}{x^2} \tag{2}$$

$$= \hat{x}\left(\frac{6Q^2}{4\pi\epsilon_0 L}\right)\left(\frac{1}{3R+L} - \frac{1}{3R}\right) \tag{3}$$

Problem 2

(a) Since the charge distribution has a high degree of symmetry, we use Gauss' law to determine the magnitude of the electric field. If we consider a cylinder of length l, radius r' and infinitesimal thickness dr', then the amount of charge inside the cylinder, and between r' and $r' + dr'$, is

$$dq = \rho(r') \cdot 2\pi l r' dr' \tag{4}$$

Therefore, the amount of charge that a cylindrical Gaussian surface of length l and radius r contains is

$$Q_{enc} = \int dq = \int_0^r \frac{2\pi\rho_0 l}{a^2} r'^3 dr' \tag{5}$$

$$= \frac{2\pi\rho_0 l}{a^2}\frac{r^4}{4} \tag{6}$$

Thus, for $r < a$, the magnitude of the electric field is given by

$$|E|(2\pi r l) = \frac{2\pi\rho_0 l}{\epsilon_0 a^2}\frac{r^4}{4} \tag{7}$$

$$|E| = \frac{\rho_0}{\epsilon_0 a^2}\frac{r^3}{4} \tag{8}$$

For $r > a$, we simply set $r = a$ in our equation for charge and find

$$|E|(2\pi r l) = \frac{2\pi\rho_0 l}{\epsilon_0}\frac{a^2}{4} \tag{9}$$

$$|E| = \frac{\rho_0}{\epsilon_0}\frac{a^2}{4r} \tag{10}$$

(b) To calculate the electric potential, we use

$$\Delta V = -\int E dr \tag{11}$$

and set $V = 0$ at r=0. For $r < a$, we get

$$V = -\frac{\rho_0}{\epsilon_0 a^2}\frac{r^4}{16} + V_0 \tag{12}$$

and for $r > a$ we have

$$V = -\frac{\rho_0 a^2}{4\epsilon_0}\ln\left(\frac{r}{r_0}\right) \tag{13}$$

To get $V(0) = 0$, we simply set $V_0 = 0$. To determine r_0, we have to ensure that V is continuous at $r = a$. Thus we require

$$-\frac{\rho_0}{\epsilon_0 a^2}\frac{a^4}{16} = -\frac{\rho_0 a^2}{4\epsilon_0}\ln\left(\frac{a}{r_0}\right) \tag{14}$$

$$r_0 = ae^{-1/4} \tag{15}$$

Problem 3

(a) We let the inner sphere have a charge Q and the outer sphere have a charge $-Q$. The electric field in between the spheres is then

$$E = \frac{1}{4\pi K\epsilon_0}\frac{Q}{r^2} \tag{16}$$

then the potential difference between the spheres is

$$\Delta V = -\int_{r_1}^{r_2} E dr \tag{17}$$

$$= \frac{Q}{4\pi K\epsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{18}$$

$$= \frac{Q}{4\pi K\epsilon_0}\left(\frac{r_1 - r_2}{r_1 r_2}\right) \tag{19}$$

Note that $|\Delta V|$ is the negative of the above since $r_1 < r_2$. Then

$$C = \frac{Q}{|\Delta V|} = 4\pi K\epsilon_0\frac{r_1 r_2}{r_2 - r_1} \tag{20}$$

(b) Following the same procedure, for the cylindrical capacitor, we have

$$E = \frac{\lambda}{2\pi K \epsilon_0 r} \tag{21}$$

so that

$$\begin{aligned} \Delta V &= -\int_{r_1}^{r_2} E dr \\ &= -\frac{\lambda}{2\pi K \epsilon_0} \ln\left(\frac{r_2}{r_1}\right) \end{aligned} \tag{22, 23}$$

Then the capacitance per unit length is

$$\frac{C}{l} = \frac{2\pi K \epsilon_0}{\ln(r_2/r_1)} \tag{24}$$

Problem 4

We consider a spherical shell of radius r and thickness dr. The resistance of such a shell is given by

$$dR = \rho \frac{dl}{A} = \frac{1}{\sigma} \frac{dr}{4\pi r^2} \tag{25}$$

so the total resistance is given by

$$R = \int_{r_1}^{r_2} \frac{1}{4\pi\sigma} \frac{dr}{r^2} = \frac{1}{4\pi\sigma}\left(\frac{1}{r_2} - \frac{1}{r_2}\right) \tag{26}$$

Problem 5

(a) Capacitors 1 and 2 are in series, so we replace them by a capacitor with capacitance

$$C_a = \frac{C_1 C_2}{C_1 + C_2} = \frac{C}{2} \tag{27}$$

The new capacitor C_a is in parallel with C_3, so we replace the two with a capacitor

$$C_b = C_3 + C_a = \frac{3C}{2} \tag{28}$$

Finally, C_4 and the new C_b are in series. The equivalent capacitance of the whole circuit is then

$$C_d = \frac{C_b C_4}{C_b + C_4} = \frac{3C}{5} \tag{29}$$

The charges on capacitors 4 and C_b is the charge on the capacitor C_d. Thus

$$Q_d = Q_4 = Q_b = \frac{3}{5}CV \tag{30}$$

$$V_4 = \frac{Q_4}{C_4} = \frac{3}{5}V \tag{31}$$

$$V_b = V_3 = V_a = \frac{2}{5}V \tag{32}$$

$$Q_3 = \frac{2}{5}CV \tag{33}$$

$$Q_a = Q_1 = Q_2 = \frac{1}{5}CV \tag{34}$$

$$V_1 = \frac{1}{5}V \tag{35}$$

$$V_2 = \frac{1}{5}V \tag{36}$$

(b) We know that $Q_1 = Q_2 = 10\mu F$. We also have

$$V_2 = \frac{Q_2}{C_2} = 5V \tag{37}$$

$$V_1 = \frac{Q_1}{C_1} = 5V \tag{38}$$

Since capacitor 3 is in parallel with 1 and 2, we have $V_3 = V_a = 10V$ and

$$Q_3 = C_3V_3 = 20\mu C \tag{39}$$

Finally, $V_b = 10V$ and $C_b = 3\mu F$, so

$$Q_4 = Q_b = V_bC_b = 30\mu C \tag{40}$$

$$V_4 = \frac{Q_4}{C_4} = 6V \tag{41}$$

Then the potential difference is

$$V_{ab} = V_4 + V_b = 6V + 10V = 16V \tag{42}$$

Problem 6 (pts.)

(a) We use the method of superposition to determine the electric field. We model the hole in the sheet as an overlap of circular disks of charge with opposite sign. The magnitude of the charge on each disk is given by $Q = \pi\sigma R^2$. Thus, the total electric field is the sum of the electric field of an infinite sheet of charge and a disk of charge $-Q$. For the disk, we first consider a ring of charge q. The electric field on the axis of the disk is given by

$$E_z = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r^2}\frac{z}{r} = \frac{q}{4\pi\epsilon_0}\frac{z}{(R^2+z^2)^{3/2}} \tag{43}$$

Where we are labelling the z axis as the axis running through the center of the ring. To get the electric field of the disk, we break the disk into infinitesimal rings of charge dq and radius r. Using the result for a ring, we have

$$E_z = \frac{z}{4\pi\epsilon_0}\int_0^R \frac{\sigma 2\pi r dr}{(r^2+z^2)^{3/2}} \tag{44}$$

$$= -\frac{z\sigma}{2\epsilon_0}\left(\frac{1}{(r^2+z^2)^{1/2}}\right)\Bigg|_0^R \tag{45}$$

$$= \frac{\sigma}{2\epsilon}\left(1 - \frac{z}{(z^2+R^2)^{1/2}}\right) \tag{46}$$

where we have used $dq = \sigma dA = \sigma 2\pi r dr = \frac{Q}{\pi R^2}\cdot 2\pi r dr$ and assumed that $z > 0$. Taking the electric field of the uniform slab, we find the total electric field is

$$\vec{E} = \hat{z}\left(\frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon}\left(1 - \frac{z}{(z^2+R^2)^{1/2}}\right)\right) = \hat{z}\left(\frac{\sigma}{2\epsilon_0}\frac{z}{(z^2+R^2)^{1/2}}\right) \tag{47}$$

(b) We now consider a position z such that $R/z \ll 1$. Then we expand

$$\frac{1}{(1+R^2/z^2)^{1/2}} = \left(1 - \frac{R^2}{2z^2}\cdots\right) = 0 + \cdots \tag{48}$$

Where we are keeping on the lowest order term. Then we see that

$$\vec{E} \sim \hat{z}\frac{\sigma}{2\epsilon_0}\left(1 - \frac{R^2}{2z^2}\right) \tag{49}$$

We are considering a point very far away from the plane. At this distance, the hole looks like a negative point charge and is the second term in the expansion. The first is the contribution from the infinite plane

(c) Now we consider $z/R \ll 1$. A similar expansion yields

$$\vec{E} \sim \hat{z}\frac{\sigma z}{2\epsilon_0 R}(1+\cdots) \tag{50}$$

As we continue to take $z \to 0$, we see the electric field drops to zero. We can look at this in two ways. If we model the hole as overlapping negative and positive charge, then as we approach the plane the two electric fields cancel. On the other hand, if we consider just the plane with a hole, if we sit at the center of the hole we see that the sum of the electric field vectors from the charges that make up the plane sum to zero.