Math 110 (Fall 2018) Midterm II (Monday October 29, 12:10-1:00)

1.

(1) (T) There exist invertible matrices E and F such that $E\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The two columns are scalar multiples of each other, so $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ has rank 1. Thus it can be transformed into $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by elementary row/column operations (Theorem 3.6). Hence there exist E and F as above (representing elementary row and column operations, respectively).

(2) (F) Let $A \in M_{n\times n}(\mathbb{R})$. If $\det(-A) = \det(A)$ then A is not invertible.

If $n = 2$ and $A = \mathrm{I}_2$ then $\det(-\mathrm{I}_2) = \det(\mathrm{I}_2) = 1$ but I_2 is invertible.

(3) (T) Let T be a linear operator on a finite-dimensional vector space V and W be a T-invariant subspace of V. If T is diagonalizable, then the characteristic polynomial of T_W splits.

Since T is diagonalizable, $\mathrm{ch}_T(t)$ splits. As $\mathrm{ch}_{T_W}(t)$ divides $\mathrm{ch}_T(t)$, we see that $\mathrm{ch}_{T_W}(t)$ must split as well.

(4) (F) Let T be a nonzero linear operator on a finite-dimensional vector space V. Let $v \in V$. Then the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.

If $V = \mathbb{R}^2$ and $T(x, y) = y$ and $v = (1, 0)$ then the T-cyclic subspace generated by v is $\mathbb{R}^2$ but the T-cyclic subspace generated by $T(v)$ is $\{0\}$.

2. (16 pts) Let A be an $m \times n$ matrix and B an $n \times p$ matrix (with entries in F). Suppose AB has rank m. Determine, with proof, the rank of A.

We know that the rank of a matrix is at most the number of its rows or columns, so in particular $\mathrm{rank}(A) \le m$. Also the rank of a product of matrices is at most the rank of each individual matrix, so in particular $\mathrm{rank}(AB) \le \mathrm{rank}(A)$. Combining these facts with the fact that $\mathrm{rank}(AB) = m$, we have

$$m = \mathrm{rank}(AB) \le \mathrm{rank}(A) \le m,$$

which implies $\mathrm{rank}(A) = m$. $\square$

Alternative proof:
Translating the statement of the problem from matrices to linear maps, we have

$$L_A : F^n \to F^m \quad \text{and} \quad L_B : F^p \to F^n \quad \text{and so} \quad L_{AB} = L_A L_B : F^p \to F^m.$$

Recall that the rank of a matrix is equal to the rank of the associated linear transformation. The rank of AB being equal to m is equivalent to L_{AB} being onto, which implies L_A is onto, and this is equivalent to A having rank m. $\square$

3. (16 pts) By any legitimate method, solve the system of linear equations over $\mathbb{R}$

$$\begin{array}{rrrrcr} x_1 & & +x_3 & +2x_4 & = & 1, \\ 2x_1 & -x_2 & +x_3 & & = & -2, \\ & x_2 & +2x_3 & +3x_4 & = & 3. \end{array}$$

(If no solution exists, explain. If there are solutions, describe the general solution.)

[See a separate note.]

4. Let $T : V \to V$ be a linear operator on a finite dimensional vector space V such that $T^2 = T$.
(1) Show that the only possible eigenvalues of T are 1 and 0.
(2) Prove that T is diagonalizable. (Hint: For every $v \in V$, show that $T(v) \in E_1$ and $v - T(v) \in E_0$. Then try to apply one of the diagonalizability criteria.)

(1) Suppose $T(v) = \lambda v$ with $\lambda \in F$, $v \neq 0$. Then

$$T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda(\lambda v) = \lambda^2 v.$$

The assumption in the problem says $T^2 = T$, so we have a chain of deductions

$$T^2(v) = T(v) \quad \Rightarrow \quad \lambda^2 v = \lambda v \quad \Rightarrow \quad (\lambda^2 - \lambda)v = 0 \quad \Rightarrow \quad \lambda^2 - \lambda = 0 \quad \Rightarrow \quad \lambda(\lambda - 1) = 0.$$

(The second last arrow is valid because $v \neq 0$.) Therefore $\lambda = 0$ or $\lambda = 1$.

(2) One criterion is that T is diagonalizable if V is spanned by the eigenspaces. In our case, by (1), it suffices to observe that V is spanned by E_0 and E_1.
For every $v \in V$, we have $T(T(v)) = T(v)$, so clearly $T(v) \in E_1$. Moreover $T(v - T(v)) = T(v) - T^2(v) = 0$ so

$$v - T(v) \in N(T) = E_0.$$

This implies that

$$v = \underbrace{(v - T(v))}_{\in\ E_0} + \underbrace{T(v)}_{\in\ E_1} \in \operatorname{span}(E_0 \cup E_1).$$

By the criterion, T is diagonalizable. $\square$

5. For the two matrices below, considered <u>over $\mathbb{R}$</u>, (1) determine whether it is diagonalizable and (2) if it is, then find an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$.

$$(a)\ A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \qquad (b)\ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

[See a separate note.]

3. (16 pts) By any legitimate method, solve the system of linear equations over $\mathbb{R}$

$$\begin{array}{rrrrcr} x_1 & & +x_3 & +2x_4 & = & 1, \\ 2x_1 & -x_2 & +x_3 & & = & -2, \\ & x_2 & +2x_3 & +3x_4 & = & 3. \end{array}$$

(If no solution exists, explain. If there are solutions, describe the general solution.)

Sol The system has the form $Ax = b$

where
$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Apply Gaussian elimination to $(A|b)$.

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 2 & -1 & 1 & 0 & -2 \\ 0 & 1 & 2 & 3 & 3 \end{array}\right) \quad \text{row } 2 - 2 \times (\text{row } 1)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & -1 & -1 & -4 & -4 \\ 0 & 1 & 2 & 3 & 3 \end{array}\right) \quad (\text{row } 2) \times (-1)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 4 & 4 \\ 0 & 1 & 2 & 3 & 3 \end{array}\right) \quad \text{row } 3 - \text{row } 2$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 4 & 4 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right) \quad \begin{array}{l} \text{row } 1 - \text{row } 3 \\ \text{row } 2 - \text{row } 3 \end{array}$$

$$\rightsquigarrow \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 5 & 5 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right) \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

This is in reduced row echelon form.

free variable. Set $x_4 = t$.

The new system of equations is

$$\begin{cases} x_1 + 3x_4 = 2 & \rightsquigarrow \quad x_1 = 2-3t \\ x_2 + 5x_4 = 5 & \rightsquigarrow \quad x_2 = 5-5t \\ x_3 - x_4 = -1 & \rightsquigarrow \quad x_3 = -1+t \end{cases}$$

$\therefore$ general sol is
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2-3t \\ 5-5t \\ -1+t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \\ 1 \\ 1 \end{pmatrix} t$$

either is correct. ■

<u>Remark</u> General sol. is not unique. Any sol of the form $s_0 + at$, where
$$\begin{cases} s_0 \text{ is sol to } Ax=b \\ a \text{ is } \text{ " } \quad Ax=0. \\ t \text{ is free variable (could have another name)} \end{cases}$$
is considered correct.

5. For the two matrices below, considered <u>over $\mathbb{R}$</u>, (1) determine whether it is diagonalizable and (2) if it is, then find an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$.

$$(a)\ A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \qquad (b)\ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

<u>Sol</u> (a) (1) $ch_A(t) = \det(A - tI) = \det\begin{pmatrix} 3-t & 2 \\ 2 & 3-t \end{pmatrix}$

$$= (3-t)^2 - 4 = t^2 - 6t + 5 = (t-1)(t-5)$$

$\therefore$ Eigenvalues are $1, 5$.

$\Rightarrow$ A is diagonalizable (since 2×2 matrix has .)

(two distinct eigenvalues)

(2) We find eigenvectors for $\lambda = 1, 5$.

$\underline{\lambda=1}$ $E_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \underbrace{(A - I)}_{= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : 2x_1 + 2x_2 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$

$\underline{\lambda=5}$ $E_5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \underbrace{(A - 5I)}_{= \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -2x_1 + 2x_2 = 0 \right\} = \operatorname{span}\left(\left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \right)$

Putting eigenvectors as columns of Q, we construct

$Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, and $Q^{-1} A Q = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

eigenvalue for $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

eigenvalue for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) (1) $ch_A(t) = \det \begin{pmatrix} 1-t & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & 0 & 1-t \end{pmatrix} = (1-t)^3.$

$\Rightarrow$ $\lambda = 1$ is the only eigenvalue, with multiplicity $m_1 = 3$.

$\dim E_1 = 3 - \operatorname{rank}(A - I) = 3 - \operatorname{rank} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Use formula
$\dim E_\lambda = n - \operatorname{rank}(A - \lambda I)$

$= 3 - 2 = 1.$

for $A: n \times n$

since $\dim E_1 < m_1$, A is not diagonalizable. $\blacksquare$