Math 110 (Fall 2018) Midterm II (Monday October 29, 12:10-1:00)

Name: SID:

Please write clearly and legibly. Justify your answers. Partial credits may be given to Problems 2, 3, 4, and 5. The last sheet of the exam is blank, for you to use for scratch work or to put solutions if necessary. If your solution is not under the problem, clearly indicate where it is. In case you have *extra scratch paper* and want it to be graded, *insert* it in your exam.

Rules during the exam:

- Do NOT start until everyone receives the exam and the proctors give the green light.
- It is NOT allowed to leave the room between 12:53–1:00, to minimize distraction for others.
- If you have a question, raise your hand and a proctor will come to you.
- If it turns out during the exam that a problem needs extra clarification, it will be put on the blackboard for everyone to see.

Tips for your solutions:

- You're free to use theorems in chapters 1–5 of the textbook without proofs.
- Facts from exercises need to be justified (unless specified otherwise by instructor).
- Using tools from outside chapters 1–5 is not encouraged, but if you utilize them: you'll get full credit if that leads to a complete solution, but little partial credit for incomplete work.
- Explain what you're doing. Connect formulas via words and sentences. Remember that you get credit only for what's written, not for what you have "actually" meant in your head.

Below some common notation is recalled.

- m, n are always positive integers,
- F is a field,
- F^n is the vector space of all n-tuples (or length n column vectors) with entries in F,
- $\{e_1, ..., e_n\}$ is the standard ordered basis for F^n,
- $M_{m\times n}(F)$ is the vector space of all $m \times n$ matrices with entries in F,
- A^t is the transpose of a matrix A,
- L_A is the left multiplication transformation of a matrix A,
- $P_n(F)$ is the vector space of polynomials with coefficients in F of degree at most n,
- $\mathrm{ch}_A(t)$, $\mathrm{ch}_T(t)$ denote characteristic polynomials of square matrix A and linear operator T,
- $E_\lambda = \{x \in F^n : Ax = \lambda x\}$ if $A \in M_{n\times n}(F)$ and $\lambda \in F$. The same symbol denotes the analogous space in the context of a linear operator T on a finite dimensional vector space V.
- m_λ denotes the multiplicity of λ when λ is an eigenvalue (of a matrix or linear operator).
- If $W \subseteq V$ is a T-invariant subspace, with $T : V \to V$ linear, then $T_W : W \to W$ is the restriction of T.

UC Berkeley Honor Code:
"As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."

1. Mark each of the following (1)–(4) True (T) or False (F). Don't give a full proof but provide a brief justification with main points, no more than a few sentences. (Correct with justification = 4 pts, Correct but no or wrong justification = 2 pts, Incorrect answer = 0 pt.) See page 1 for notation. In problem 1, every matrix or vector space is considered over $\mathbb{R}$, the field of real numbers.

(1) (   ) There exist invertible matrices E and F such that $E\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(2) (   ) Let $A \in M_{n\times n}(\mathbb{R})$. If $\det(-A) = \det(A)$ then A is not invertible.

(3) (   ) Let T be a linear operator on a finite-dimensional vector space V and W be a T-invariant subspace of V. If T is diagonalizable, then the characteristic polynomial of T_W splits.

(4) (   ) Let T be a nonzero linear operator on a finite-dimensional vector space V. Let $v \in V$. Then the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $T(v)$.

2. (16 pts) Let A be an $m \times n$ matrix and B an $n \times p$ matrix (with entries in F). Suppose AB has rank m. Determine, with proof, the rank of A.

3. (16 pts) By any legitimate method, solve the system of linear equations over $\mathbb{R}$

$$\begin{array}{rrrrcr} x_1 & & +x_3 & +2x_4 & = & 1, \\ 2x_1 & -x_2 & +x_3 & & = & -2, \\ & x_2 & +2x_3 & +3x_4 & = & 3. \end{array}$$

(If no solution exists, explain. If there are solutions, describe the general solution.)

4. Let $T : V \to V$ be a linear operator on a finite dimensional vector space V such that $T^2 = T$.

(1) Show that the only possible eigenvalues of T are 1 and 0.

(2) Prove that T is diagonalizable. (Hint: For every $v \in V$, show that $T(v) \in E_1$ and $v - T(v) \in E_0$. Then try to apply one of the diagonalizability criteria.)

5. For the two matrices below, considered over $\mathbb{R}$, (1) determine whether it is diagonalizable and (2) if it is, then find an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$.

$$(a)\ A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \qquad (b)\ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$