Problem 1 (50 points)

(a) By pattern-matching the coefficients,

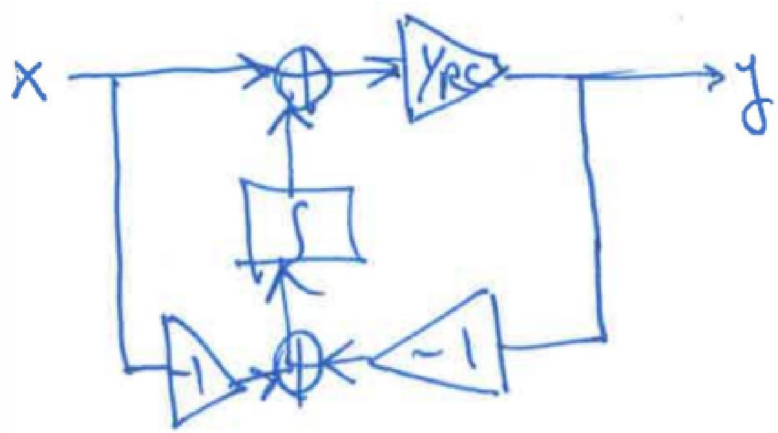
$$(i\omega RC + 1)H(\omega) = i\omega RC - 1 \implies RC\dot{y} + y = RC\dot{x} - x$$

Giving us the desired linear, constant-coefficient differential equation.

$$\dot{y} = -\frac{1}{RC}y + \dot{x} - \frac{1}{RC}x$$

In creating the integrator-adder-gain block diagram implementation of the filter, it will be illustrative to integrate the differential equation:

$$y = -\frac{1}{RC}\int y + x - \frac{1}{RC}\int x$$



(b)

$$H(\omega) = \frac{i\omega RC - 1}{i\omega RC + 1} = \frac{i\omega - \frac{1}{RC}}{i\omega + \frac{1}{RC}} = -\frac{\frac{1}{RC} - i\omega}{\frac{1}{RC} + i\omega}$$

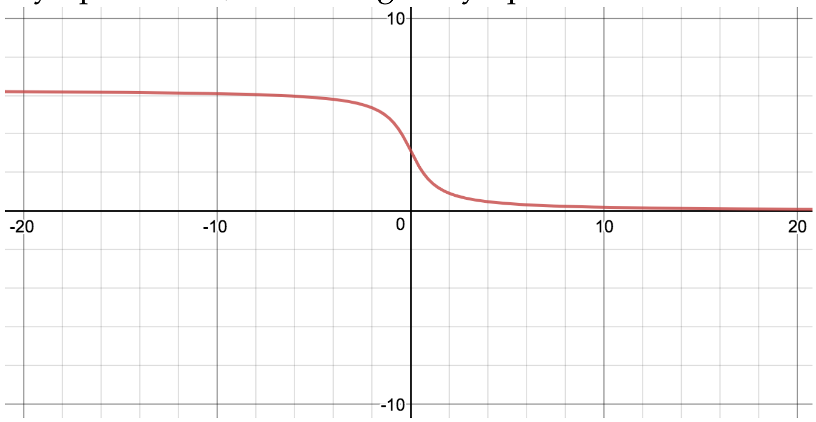
Observing that the numerator is the complex conjugate of the denominator, we find that

$$|H(\omega)| = 1$$

This is an all-pass filter. Solving for the phase response, we have

$$\begin{aligned}\angle H(\omega) &= \angle(-1) + \angle\left(\frac{1}{RC} - i\omega\right) - \angle\left(\frac{1}{RC} + i\omega\right) \\ &= \pi - 2\tan^{-1}(RC\omega)\end{aligned}$$

The plot is as follows, where the x-axis is ω, the y-axis is $\angle H(\omega)$, the y-intercept is π, the left asymptote is 2π, and the right asymptote is 0.



(c)

$$G(\omega) = \int_{-\infty}^{\infty} \frac{\mathrm{d}f(t)}{\mathrm{d}t} e^{-i\omega t}\mathrm{d}t = \int_{-\infty}^{\infty} e^{-i\omega t}\mathrm{d}f(t)$$

Using integration by parts, where $u = e^{-i\omega t}$ and $\mathrm{d}v = \mathrm{d}f(t)$, such that $\mathrm{d}u = -i\omega e^{-i\omega t}\mathrm{d}t$ and $v = f(t)$, we have

$$G(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\mathrm{d}f(t) = f(t)e^{-i\omega t}\Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t = 0 + i\omega F(\omega)$$

Where the first term goes to 0 because we assume $\lim\limits_{t\to\pm\infty} f(t) = 0$, a clarification made during the examination. The second term can be recognized from the analysis equation, with a coefficient of $i\omega$ falling out of the $\mathrm{d}u$ term in integration by parts.

(d)

$$H(\omega) = \frac{i\omega - \frac{1}{RC}}{i\omega + \frac{1}{RC}} = i\omega\left(\frac{1}{i\omega + \frac{1}{RC}}\right) - \frac{1}{RC}\frac{1}{i\omega + \frac{1}{RC}}$$

Observing the Fourier pair

$$e^{-\frac{t}{RC}}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{i\omega + \frac{1}{RC}}$$

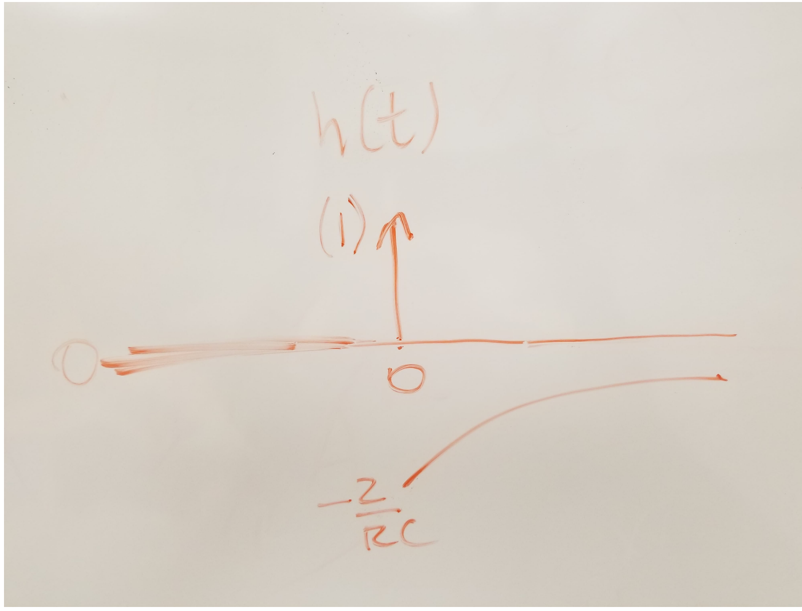
We know that

$$-\frac{1}{RC}1e^{-\frac{t}{RC}}u(t) \overset{\mathcal{F}}{\longleftrightarrow} -\frac{1}{RC}\frac{1}{i\omega + \frac{1}{RC}}$$

Using part (c), we find that

$$i\omega\left(\frac{1}{i\omega + \frac{1}{RC}}\right) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{t}{RC}}u(t)\right) = \delta(t) - \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

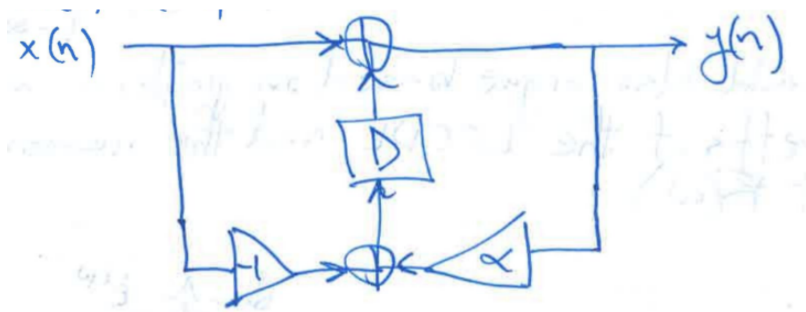
Hence $H(\omega)$ must be the frequency response of $h(t) = \delta(t) - \frac{2}{RC}e^{-\frac{t}{RC}}u(t)$.



Problem 2 (60 points)

(a)

$$y(n) = \alpha y(n-1) + x(n) - x(n-1)$$



(b) We present two approaches.
Method 1:

$$\begin{aligned}
f(n) &= \alpha f(n-1) + \delta(n) - \delta(n-1) \\
f(0) &= \alpha f(-1) + \delta(0) - \delta(-1) = 1 \\
f(1) &= \alpha f(0) + \delta(1) - \delta(0) = \alpha - 1 \\
f(2) &= \alpha f(1) = \alpha(\alpha - 1) \\
f(3) &= \alpha f(2) = \alpha^2(\alpha - 1) \\
&\vdots \\
f(n) &= \alpha^{n-1}(\alpha - 1)
\end{aligned}$$

$$f(n) = \alpha^n u(n) - \alpha^{n-1}u(n-1) = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \alpha^{n-1}(\alpha - 1) & n \geq 1 \end{cases}$$

Method 2:

$$F(\omega) = \frac{1 - e^{-i\omega}}{1 - \alpha e^{-i\omega}} = \frac{1}{1 - \alpha e^{-i\omega}} - \frac{e^{-i\omega}}{1 - \alpha e^{-i\omega}}$$

Using the time-shift property of the Fourier transform and the Fourier pair

$$\frac{1}{1 - \alpha e^{-i\omega}} \overset{\mathcal{F}}{\leftrightarrow} \alpha^n u(n)$$

We have

$$f(n) = \alpha^n u(n) - \alpha^{n-1}u(n-1)$$

(c) Passing in input $x(n) = e^{i\omega n}$ into the given LCCDE, we obtain

$$F(\omega)e^{i\omega n} - \alpha e^{-i\omega}F(\omega)e^{i\omega n} = e^{i\omega n} - e^{-i\omega}e^{i\omega n}$$

Dividing by the complex exponential and grouping terms, we have

$$\left(1-\alpha e^{-i\omega}\right) F(\omega) = 1-e^{-i\omega} \implies F(\omega) = \frac{1-e^{-i\omega}}{1-\alpha e^{-i\omega}}$$

(d)

$$F(\omega) = \frac{e^{i\omega}-1}{e^{i\omega}-0.99}$$

If α is close to 1, as is the case here, the two vectors representing $e^{i\omega}-1$ and $e^{i\omega}-\alpha$ are nearly identical for $e^{i\omega}$ not in the close neighborhood of 1, so we find that $|F(\omega)| \approx 1$. This is a notch filter with notch at $\omega = 0$.

(e)

(i)

$$x(n) = \frac{1}{2}e^{i0n} + \frac{1}{2}e^{i\pi n} \implies y(n) = \frac{F(0)}{2}e^{i0n} + \frac{F(\pi)}{2}e^{i\pi n} = \frac{F(\pi)}{2}e^{i\pi n}$$

From the expression in the previous part, we know $F(\pi) = \frac{2}{1+\alpha}$, so

$$y(n) = \frac{(-1)^n}{1+\alpha}$$

(ii)

$$\alpha = 0.99 \implies F(\pi) \approx 1 \implies y(n) \approx \frac{(-1)^n}{2}$$

The plot is an alternating sequence of $\pm\frac{1}{2}$, with $y(0) = \frac{1}{2}$.

Problem 3 We know that when $y(n) = \delta(n)$, $r(n) = g(n)$. If we can find an input $\hat{x}(n)$ such that the corresponding $\hat{y}(n) = \delta(n)$, then this $\hat{x}(n) = g(n)$, since $x(n) = r(n)$. From plotting the given input-output pair, we can observe that $y(n) - \alpha y(n-N) = \delta(n)$. Since F is an LTI system, $\hat{x}(n) = x(n) - \alpha x(n-N)$ is the impulse response $g(n)$ of the system G. That is,

$$g(n) = \delta(n) + \delta(n-1) - \alpha\delta(n-N) - \alpha\delta(n-N-1)$$

And so we can read off the frequency response as

$$G(\omega) = 1 + e^{-i\omega} - \alpha e^{-i\omega N} - \alpha e^{-i\omega(N+1)}$$

Problem 4 (50 points)

(a)

$$y(t) = \int_{-\infty}^{\infty} h(\sigma)x(t-\sigma)\mathrm{d}\sigma = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-\sigma)\mathrm{d}\sigma$$

By a change of variables $\tau = t - \sigma$, such that $\sigma = t - \tau$ and $\mathrm{d}\sigma = -\mathrm{d}\tau$,

$$= \frac{1}{T}\int_{t+\frac{T}{2}}^{t-\frac{T}{2}} x(\tau)(-\mathrm{d}\tau) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)\mathrm{d}\tau$$

Then we can read off the values

$$A = \frac{T}{2}$$

$$B = T$$

This is a moving average filter over a time window T.

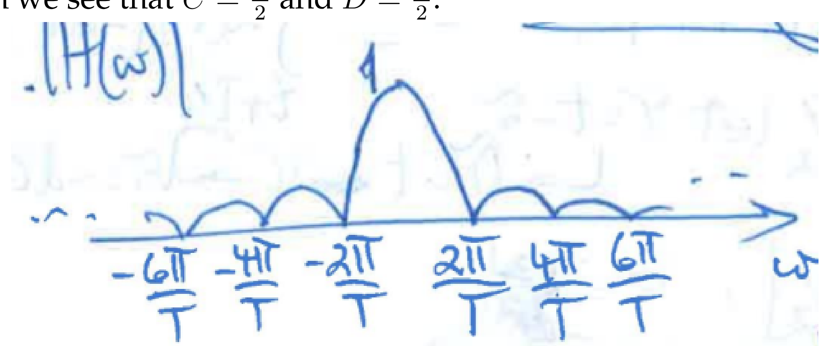
(b) Observing that $\lim_{T\to 0} h(t) = \delta(t)$, the system approaches the identity system.

$$\lim_{T\to 0} y(t) = x(t)$$

(c)

$$\begin{aligned} H(\omega) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\omega t}\mathrm{d}t \\ &= \frac{1}{T}\frac{e^{-i\omega}}{-i\omega}\bigg|_{-\frac{T}{2}}^{\frac{T}{2}} \\ &= \frac{e^{\frac{i\omega T}{2}} - e^{-\frac{i\omega T}{2}}}{i\omega T} \\ &= \frac{2}{\omega T}\frac{e^{\frac{i\omega T}{2}} - e^{-\frac{i\omega T}{2}}}{2i} \\ &= \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} \end{aligned}$$

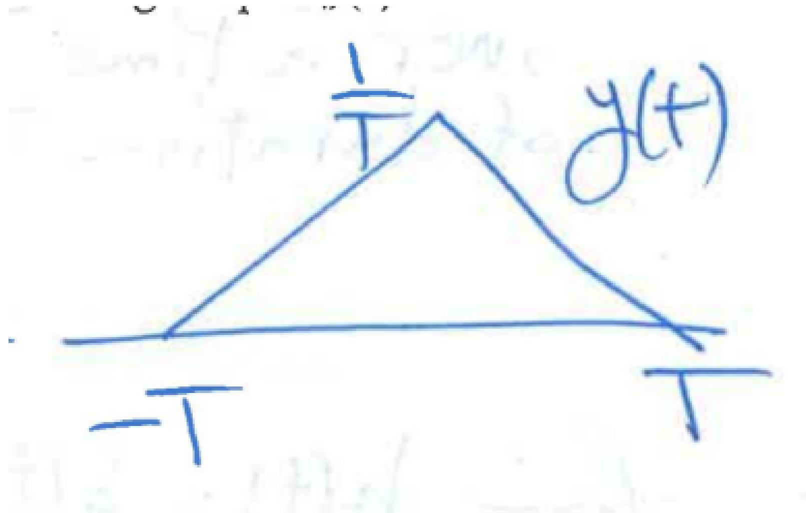
From which we see that $C = \frac{T}{2}$ and $D = \frac{T}{2}$.



(d)

(i)

$$y(t) = \begin{cases} \frac{T+t}{T^2} & -T \le t \le 0 \\ \frac{T-t}{T^2} & 0 < t \le T \\ 0 & |t| > T \end{cases}$$



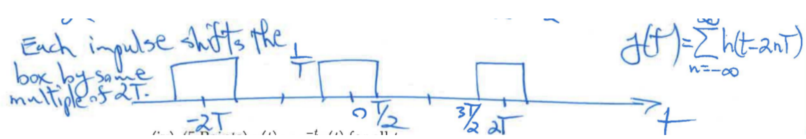
(ii)

$$\omega_0 = \frac{4\pi}{T} \implies T_0 = \frac{2\pi}{\omega_0} = \frac{T}{2}$$

Since the cosine undergoes two cycles over a duration of T,

$$y(t) = 0$$

(iii)



(iv)

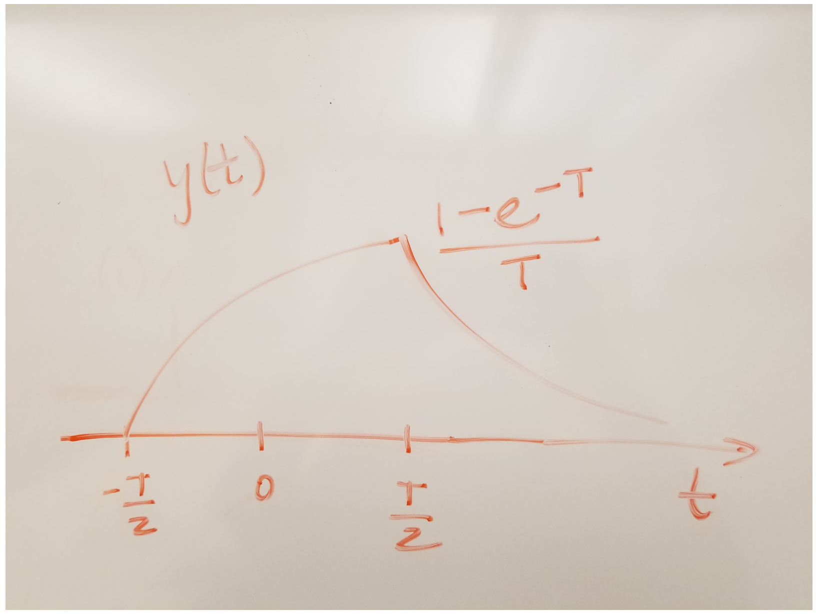
$$y(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)\mathrm{d}\tau$$

Case 1: $t < -\frac{T}{2} \implies y(t) = 0$
Case 2: $-\frac{T}{2} \le t \le \frac{T}{2}$

$$y(t) = \frac{1}{T}\int_{0}^{t+\frac{T}{2}} e^{-\tau}\mathrm{d}\tau = -\frac{e^{-\tau}}{T}\bigg|_0^{t+\frac{T}{2}} = \frac{1-e^{-\left(t+\frac{T}{2}\right)}}{T}$$

Case 3: $t > \frac{T}{2}$

$$y(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} e^{-\tau}\mathrm{d}\tau = -\frac{1}{T}e^{-\tau}\bigg|_{t-\frac{T}{2}}^{t+\frac{T}{2}} = \frac{e^{-\left(t-\frac{T}{2}\right)} - e^{-\left(t+\frac{T}{2}\right)}}{T} = e^{-T}\left(\frac{e^{\frac{T}{2}} - e^{-\frac{T}{2}}}{T}\right)$$


y(t)
1−e−T
T
-T
2
0
T
2
t