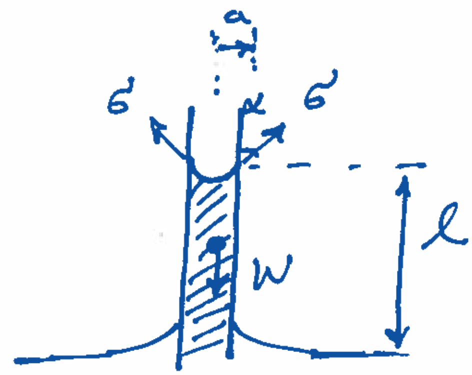
ME 106 | Midterm 1 - Solutions

• Problem 1

(a) for the water rised in the capillary tube:



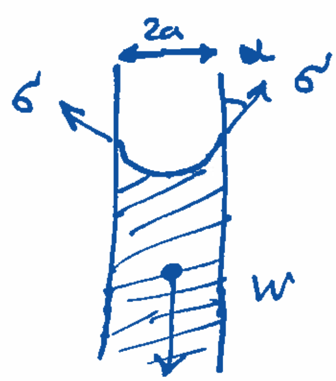
$W =$ surface tension force

$\rho \forall g = \rho(\pi a^2)\ell g$ $\qquad 2\pi a \sigma \cos\alpha$

$\Rightarrow \quad \rho(\cancel{\pi} a^{2})\ell g = 2\cancel{\pi} a \sigma \cos\alpha$

$\Rightarrow \quad \ell = \dfrac{2\sigma\cos\alpha}{\rho g a} \Rightarrow \boxed{\ell = \dfrac{2\sigma\cos\alpha}{\gamma a}}$

(b) Let's assume the width (normal to the page) is some "b".



Now: $\quad W =$ surface tension force

$\rho \forall g = \rho(2a)(b)\ell g \qquad 2 \times (\sigma \cos\alpha) b$

$$\Rightarrow \quad \rho(2ab)\,l\,g = 2\sigma\cos\alpha\, b$$

$$\Rightarrow \quad l = \frac{\cancel{2}\sigma\cos\alpha\,\cancel{b}}{\rho g\,(\cancel{2}a\cancel{b})} \Rightarrow \boxed{l = \frac{\sigma\cos\alpha}{\gamma a}}$$

Capillary rise will be half of the one in case a.

• Problem 2

• Given: flow field $\begin{cases} u = \dfrac{x}{1+2t} \\ v = \dfrac{y}{2t} \end{cases}$

particle passing through (x_0, y_0) at $t = t_0$.

Ans:

• Flow field is 2D & Unsteady $\left(\frac{\partial u}{\partial t} \neq 0 \;\&\; \frac{\partial v}{\partial t} \neq 0\right)$

• let's first find the SL:

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{x/(1+2t)} = \frac{dy}{y/2t} \Rightarrow$$

$$\Rightarrow (1+2t)\frac{dx}{x} = 2t\frac{dy}{y} \Rightarrow \frac{1+2t}{2t}\int_{x_0}^{x}\frac{dx}{x} = \int_{y_0}^{y}\frac{dy}{y} \Rightarrow$$

$$\Rightarrow \quad \frac{1+2t}{2t} \ln(x/x_0) = \ln(y/y_0) \Rightarrow$$

$$\Rightarrow \ln\left[(x/x_0)^{\frac{1+2t}{2t}}\right] = \ln y/y_0 \Rightarrow \boxed{y/y_0 = (x/x_0)^{\frac{1+2t}{2t}}}$$

• Now let's find the pathline of the particle:

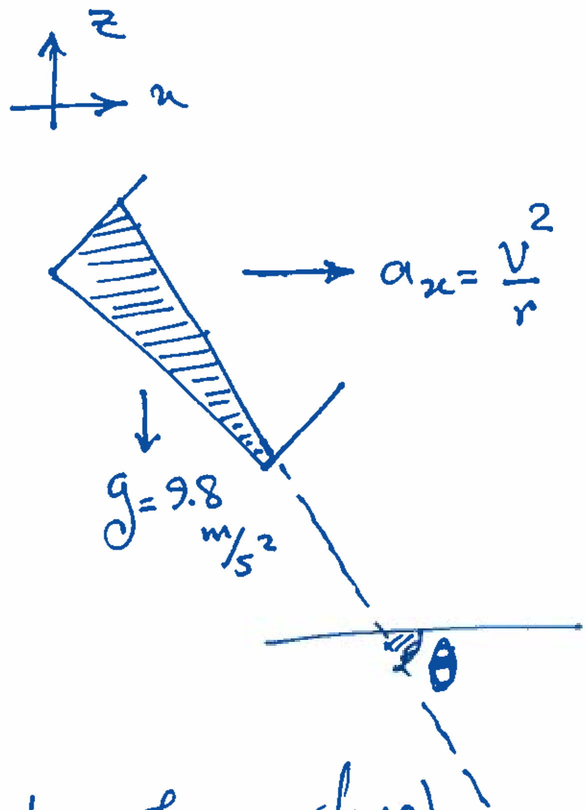
$$\rightarrow \frac{dx}{dt} = u = \frac{x}{1+2t} \Rightarrow \int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} \frac{dt}{1+2t} \Rightarrow$$

$$\Rightarrow \ln(x/x_0) = \frac{1}{2} \ln\left(\frac{1+2t}{1+2t_0}\right) \Rightarrow \boxed{x/x_0 = \left(\frac{1+2t}{1+2t_0}\right)^{1/2}}$$

$$\rightarrow \frac{dy}{dt} = v = \frac{y}{2t} \Rightarrow \int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} \frac{dt}{2t} \Rightarrow$$

$$\Rightarrow \ln y/y_0 = \frac{1}{2} \ln(t/t_0) \Rightarrow \boxed{y/y_0 = (t/t_0)^{1/2}}$$

1. Problem 3



$V = 180\ km/h = 50\ m/s$

$\Rightarrow a_x = \frac{50^2}{250} = 10\ m/s^2$

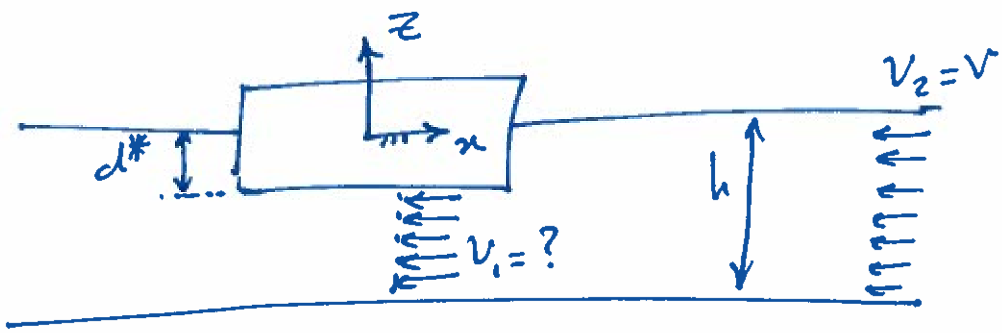
Now for the contours of constant pressures (including free surface):

$$\frac{dz}{dx} = \frac{-a_x}{g} = \frac{-10}{9.8} \simeq 1$$

$$\theta = tg^{-1}\left(-\frac{dz}{dx}\right) = tg^{-1}(1) \approx \boxed{45^\circ}$$

• Problem 4

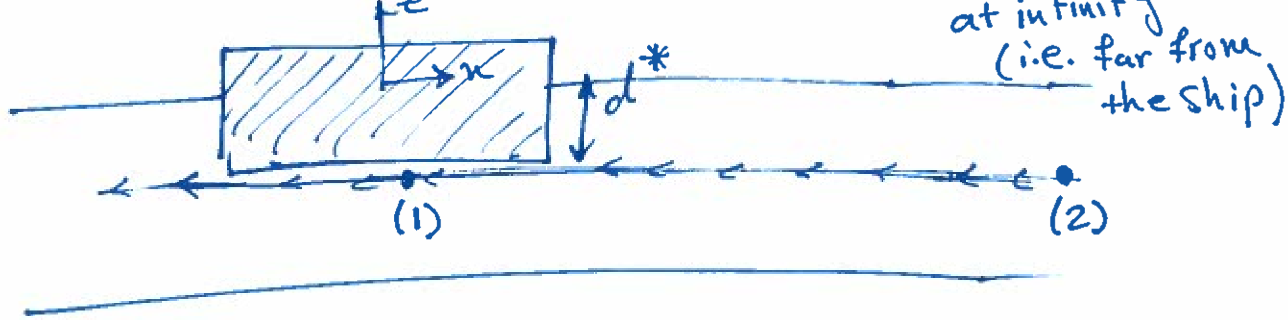
Let's assume the ship is cruising with the new draft 'd^*'. Then for the frame of reference fixed to the ship:



from conservation of mass:

$$h V_2 = (h - d^*) V_1 \Rightarrow V_1 = \frac{h}{h - d^*} V \quad \text{(I)}$$

Now let's consider the streamline right below the ship:



Note: Point (2) is picked far from the ship!

Using the Bernoulli's eq. for (1) & (2):

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + \cancel{g z_1} = \frac{P_2}{\rho} + V_2^2/2 + \cancel{g z_2}$$

far from the ship

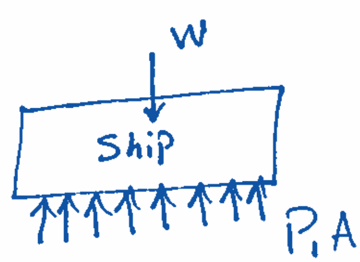
Note: $z_1 = z_2$; $V_2 = V$; $V_1 = \frac{h}{h-d^*} V$; $P_2 = \rho g d^*$

Substituting into the above eq.:

$$\frac{P_1}{\rho} = \frac{\rho g d^*}{\rho} + \frac{V^2}{2} - \frac{1}{2}\left(\frac{h}{h-d^*} V\right)^2$$

$$\Rightarrow \quad P_1/\rho = g d^* + \frac{1}{2}\left(1 - \left(\frac{h}{h-d^*}\right)^2\right) V^2 \qquad \text{II}$$

We also know that:



$$\underbrace{P_1 A}_{\text{After cruising}} = W_{ship} = \underbrace{(\rho g d) A}_{\text{before cruising}}$$

$$\Rightarrow \quad P_1/\rho = g d \xrightarrow{\text{from II}} g d^* + \frac{1}{2}\left[1 - \left(\frac{h}{h-d^*}\right)^2\right] V^2 = g d$$

$$\Rightarrow \quad \boxed{d^* = d + \frac{V^2}{2g}\left[\left(\frac{h}{h-d^*}\right)^2 - 1\right]}$$

"unknown"

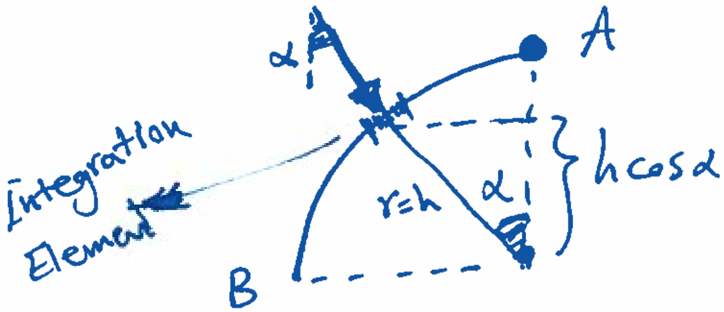
d^* can be found from this Eq.

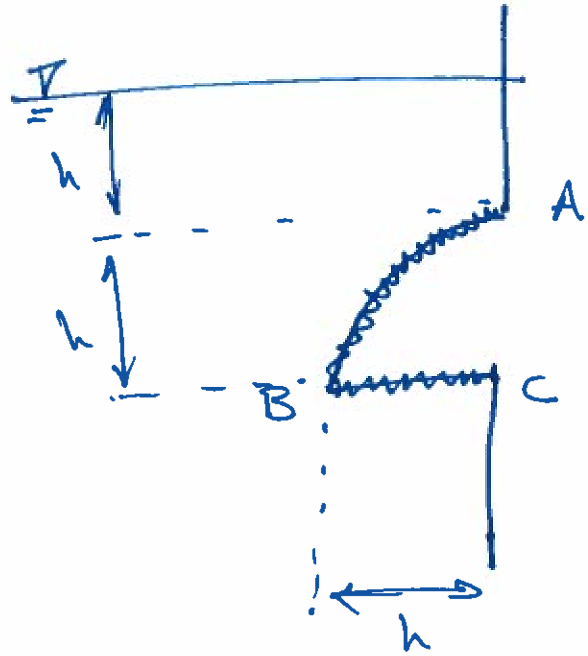
$\boxed{\text{• Problem } \underline{5}}$

width normal to page $=$ 'b'

a

$P = \gamma\left[h + (h - h\cos\alpha)\right]$





horizontal force on the element:

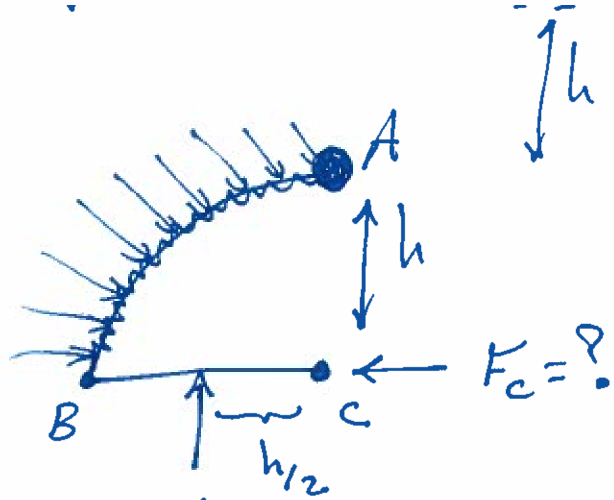
$$dF_x = \underbrace{P}_{\gamma\left[h+(h-h\cos\alpha)\right]}\underbrace{dA}_{h\,d\alpha\, b}\sin\alpha = \underbrace{\gamma h(2-\cos\alpha)}\,\underbrace{hb\,d\alpha}\,\underbrace{\sin\alpha}$$

$$\Rightarrow F_x = \int_{\alpha=0}^{\alpha=\pi/2} dF_x = \int_{\alpha=0}^{\alpha=\pi/2} P\,dA\sin\alpha = \int_0^{\pi/2}\gamma h(2-\cos\alpha)\,hb\sin\alpha\,d\alpha =$$

$$= \gamma b h^2\int_0^{\pi/2}(2\sin\alpha - \underbrace{\sin\alpha\cos\alpha}_{=\frac{1}{2}\sin 2\alpha})\,d\alpha = \gamma b h^2\underbrace{\left(-2\cos\alpha + \frac{1}{4}\cos 2\alpha\right)\Big|_0^{\pi/2}}_{=3/2}$$

$$\Rightarrow \boxed{F_x = \frac{3}{2}\gamma b h^2}$$

(b)



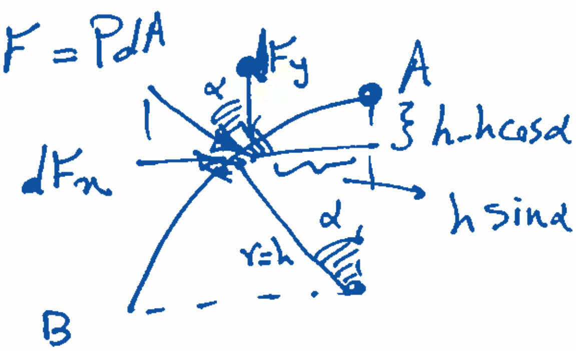
$$\sum M_A = 0$$

$$\Downarrow$$

$$\boxed{\underbrace{M_{AB}}_{CCW} - \underbrace{F_{BC} \times h/2}_{CW} - \underbrace{F_c \times h}_{CW} = 0} \quad (*)$$

$$F_{BC} = \gamma(h+h) \times A_{BC} =$$
$$= 2\gamma h \times (bh) = 2\gamma b h^2$$

Therefore, we need to find moment contribution of force distribution on AB:



$$M_{AB} = \int dF_x (h - h\cos\alpha) + dF_y (h\sin\alpha) =$$

$$= \int_{\alpha=0}^{\alpha=\frac{\pi}{2}} \underbrace{\gamma h (2-\cos\alpha) hb\, d\alpha \sin\alpha}_{\text{from (a): } dF_x} \underbrace{(h - h\cos\alpha)} +$$

$$+ \underbrace{\gamma h (2-\cos\alpha) hb\, d\alpha \cos\alpha}_{dF_y} \underbrace{(h\sin\alpha)} \Rightarrow$$

$$\Rightarrow M_{AB} = \gamma h^3 b \int_0^{\frac{\pi}{2}} (2-\cos\alpha)(1-\cos\alpha)\sin\alpha \, d\alpha$$

$$+ \gamma h^3 b \int_0^{\frac{\pi}{2}} (2-\cos\alpha)\cos\alpha \sin\alpha \, d\alpha$$

let's define $\Rightarrow$ $u = \cos\alpha \rightarrow du = +\sin\alpha \, d\alpha$; $\alpha = 0 \rightarrow u = 1$; $\alpha = \pi/2 \rightarrow u = 0$

$$\Rightarrow M_{AB} = \gamma b h^3 \int_{u=-1}^{0} + (2+u)(1+u) \, du +$$

$$+ \gamma b h^3 \int_{u=-1}^{0} (2+u)(-u) \, du =$$

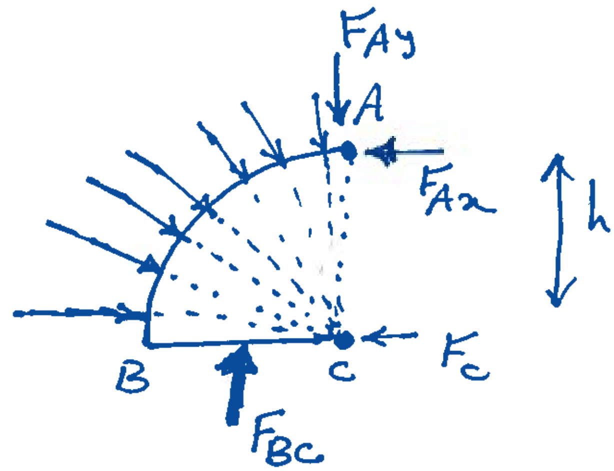
$$= \gamma b h^3 \left[\left(u^3/3 + 3u^2/2 + 2u \right|_{-1}^{0} + \left(-u^3/3 - u^2 \right|_{-1}^{0} \right] =$$

$$= \gamma b h^3 \left(5/6 + 2/3 \right) = \boxed{9/6 \, \gamma b h^3}$$

Now let's substitute M_{AB} into ⊛ :

$$\underbrace{M_{AB}}_{9/6\,\gamma b h^3} = \underbrace{F_{BC}}_{2\gamma b h^2} \times h/2 + F_C \times h \Rightarrow \boxed{F_C / \gamma b h^2 = 1/2}$$

* A more clever way to solve problem 5b:



Note: pressure force is always normal to the surface, thus the force distribution on AB does not have any moment contribution around 'C' → because all the force element are normal to AB (the line of action for them passes through C)

Now: Using static

$$\Sigma M_C = 0 \Rightarrow F_{Ax} \times h - F_{BC} \times \frac{h}{2} = 0$$

$$\Rightarrow F_{Ax} = F_{BC}/2 = [\gamma(h+h)(bh)]/2 = \gamma b h^2 \quad (*)$$

Then let's use balance of force in x direction:

$$\Sigma F_x = 0 \Rightarrow \underbrace{-F_x}_{\text{Contribution of force distribution of AB; we found it in part (a)}} + \underbrace{F_{Ax}}_{\gamma b h^2 \text{ (from } (*))} + F_C = 0 \Rightarrow \boxed{F_C = \frac{1}{2}\gamma b h^2}$$

from (a) → $F_x = 3/2\, \gamma b h^2$