Math 1A (Fall 2017) Final Exam Solutions (Friday December 15, 19:10-22:00)

1. Mark each of the following True (T) or False (F). No justification is necessary.
(For each sub-problem, correct = 4 pts, no response = 2 pts, wrong = 0 pts.)

(1) (F) Suppose that $f(g(x))$ is continuous at 0. Then f is continuous at $g(0)$, and g is continuous at 0.

This is false. For instance: If $f(x) = 1$ (constant function) and g is any function that is not continuous at 0 then $f(g(x))$ is the constant function 1 so continuous at 0.

(2) (F) If $f(x) = -\cos x$ then its 31st derivative $f^{(31)}(x) = \sin x$.

$f(x) = f^{(4)}(x) = f^{(8)}(x) = \cdots = f^{(28)}(x) = -\cos x$. So $f^{(31)}(x) = f^{(3)}(x) = -\sin x$. (See Stewart Section 3.3, Example 4 for a similar computation.)

(3) (T) Let a be a constant. Let f and g be functions that are continuous everywhere. If $\int_a^x f(t)dt = \int_a^x g(t)dt$ as functions of x, then $f(x) = g(x)$.

Applying d/dx and FTC1, we obtain $f(x) = g(x)$ indeed.

(4) (F) For any continuous function f and any real numbers a and b,

$$\int_a^b |f(x)|\, dx = \left| \int_a^b f(x)\, dx \right|.$$

In general only the inequality holds that the left hand side is greater than or equal to the right hand side. Note that if $f(x) = x$ then $|x|$ is an even function so

$$\int_{-1}^{1} |x| dx = 2 \int_0^1 x dx = x^2 \Big]_0^1 = 1$$

but x is an odd function and $\int_{-1}^{1} x dx = 0$.

(5) (T) If the functions f and g have antiderivatives F and G, respectively, then FG is an antiderivative of the function $Fg + fG$.

Indeed by the product rule, $(FG)' = F'G + FG' = fG + Fg$.

(6) (F) Suppose that $\lim_{x \to 0} f(x) = 1$. Then for every number $\epsilon > 0$, there is a number $\delta > 0$ such that if $|x| < \delta$ then $|f(x) - 1| < \epsilon$.

The correct condition should have $0 < |x| < \delta$ in place of $|x| < \delta$. In other words, the inequality $|f(x) - 1| < \epsilon$ need not hold for $x = 0$. Even if $\lim_{x \to 0} f(x) = 1$, the value of $f(0)$ may be very far from 1.

(7) (T) If $f(x)$ is an even continuous function then $\int_0^x f(t)dt$ is an odd function.

This can be seen either intuitively or from the substitution rule. Put $F(x) = \int_0^x f(t)dt$. The latter approach tells us that by substituting $u = -t$ thus $du = -dt$,

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^x -f(-u)du.$$

Since f is even, $f(-u) = f(u)$ so $\int_0^x -f(-u)du = -\int_0^x f(u)du = -F(x)$. Therefore F is odd.

(8) (T) Suppose that both f is differentiable twice and that f'' is continuous on their domains. If f has an inflection point then f' has a local maximum or a local minimum.

Observe that f'' changes sign either from $-$ to $+$ or from $+$ to $-$ at an inflection point. By the first derivative test applied to f', this means that f' has a local maximum or a local minimum at that point.

(9) (F) If the line $x = 1$ is a vertical asymptote of $y = f(x)$ then f is not defined at 1.

A vertical asymptote at 1 only has to do with (one-sided) limits as x approaches 1. It has nothing to do with whether f is defined at 1 or what value f takes at 1. For instance if you define $f(x)$ to be $1/(x-1)$ when $x \neq 1$ and 0 when $x = 1$, then $y = f(x)$ has a vertical asymptote at $x = 1$ but still f is defined at 1.

(10) (F) Let f and g be functions. Then the domain of the composite function $f \circ g$ is contained in the domain of f.

This is false. For instance if $f(x) = 1/x$ and $g(x) = x - 1$ then f has domain $\{x \neq 0\}$ but $f \circ g$ has domain $\{x \neq 1\}$. (In general one only knows that the domain of the composite function $f \circ g$ is contained in the domain of g.)

2. (1) (5 points) $\lim_{x \to 0^+} (1 + 2x)^{1/x}$

Let $f(x) = (1 + 2x)^{1/x}$. Then

$$\ln f(x) = \frac{\ln(1 + 2x)}{x}.$$

This is indeterminate of the form $0/0$ as $x \to 0^+$. Applying l'Hospital's rule,

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln(1 + 2x)}{x} = \lim_{x \to 0^+} \frac{\frac{2}{1+2x}}{1} = 2.$$

Therefore $\lim_{x \to 0^+} f(x) = e^{\lim_{x \to 0^+} \ln f(x)} = \boxed{e^2}$.

(2) (5 points) $\int_{\pi^2/9}^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

We substitute $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$. When $x = \pi^2/9$ and $x = \pi^2/4$, we have $u = \pi/3$ and $u = \pi/2$, resp. Hence

$$\int_{\pi^2/9}^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}} dx = \int_{\pi/3}^{\pi/2} 2\sin u du = -2\cos u\Big]_{\pi/3}^{\pi/2} = -2(\cos\frac{\pi}{2} - \cos\frac{\pi}{3}) = -2(-\frac{1}{2}) = \boxed{1}.$$

(3) (5 points) the second derivative of e^{e^x}

By the chain rule, $(e^{e^x})' = (e^x)'e^{e^x} = e^x e^{e^x} = e^{e^x + x}$.

Differentiating again, $(e^{e^x})'' = (e^{e^x+x})' = (e^x + x)'e^{e^x+x} = \boxed{(e^x + 1)e^{e^x+x}}$.

(4) (5 points) horizontal asymptotes of $y = \dfrac{3e^x}{e^x - 2}$.

$$\lim_{x\to\infty} \frac{3e^x}{e^x - 2} = \lim_{x\to\infty} \frac{3}{1 - 2e^{-x}} = \frac{3}{1 - 2\cdot 0} = 3.$$
$$\lim_{x\to-\infty} \frac{3e^x}{e^x - 2} = \frac{3\cdot 0}{0 - 2} = 0.$$

Therefore $\boxed{y = 3 \text{ and } y = 0}$ are horizontal asymptotes.

(Note: The first limit can also be computed by l'Hospital's rule, but not the second. The second limit is not of the form $0/0$ or ∞/∞.)

3. (1) (10 points) The velocity function of a particle is $v(t) = \dfrac{\sin t}{1 + \cos^2 t}$ m/s, where t stands for time in seconds. Find the total distance (*not displacement*) the particle traveled from time $t = 0$ to $t = 2\pi$.

Since the denominator is positive, $v(t)$ has the same sign as $\sin t$. So $v(t) \geq 0$ if $0 \leq t \leq \pi$ and $v(t) \leq 0$ if $\pi \leq t \leq 2\pi$. Hence the total distance is

$$\int_0^{2\pi} |v(t)| dt = \int_0^{\pi} v(t)dt + \int_{\pi}^{2\pi} -v(t)dt = \int_0^{\pi} v(t)dt - \int_{\pi}^{2\pi} v(t)dt.$$

Substituting $u = \cos t$, we have $du = -\sin t dt$ and $v(t)dt$ turns into $-\frac{1}{1+u^2}du$. Hence

$$\int_0^{\pi} v(t)dt - \int_{\pi}^{2\pi} v(t)dt = \int_1^{-1} -\frac{1}{1+u^2}du - \int_{-1}^{1} -\frac{1}{1+u^2}du = \int_{-1}^{1} \frac{1}{1+u^2}du + \int_{-1}^{1} \frac{1}{1+u^2}du$$
$$= 2\int_{-1}^{1} \frac{1}{1+u^2}du = 2\tan^{-1} u\Big]_{-1}^{1} = 2(\frac{\pi}{4} - (-\frac{\pi}{4})) = \boxed{\pi}.$$

(2) (10 points) Find y' in terms of x and y if $y = \ln(xy + x^2)$.

By implicit differentiation,

$$y' = \frac{(xy + x^2)'}{xy + x^2}.$$

We have $(xy + x^2)' = (xy)' + (x^2)' = x'y + xy' + 2x = y + xy' + 2x$. (All differentiations are with respect to x.) Hence

$$y' = \frac{y + xy' + 2x}{xy + x^2}.$$
$$y'(xy + x^2) = y + xy' + 2x.$$
$$y'(xy + x^2 - x) = y + 2x.$$
$$\boxed{y' = \frac{y + 2x}{xy + x^2 - x}}.$$

(You could factor out x in the denominator.)

4. (20 points) A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out? (The person stays in the same place. The string always makes a straight line.)

Let $f(t)$ be the horizontal distance between the person and the kite as a function of time t. Let $\theta(t)$ denote the angle in the question. We are given $f'(t) = 8$. If t_0 is the moment when the length of the string is 200, the problem asks to find $\theta'(t_0)$.

This is a relates rates problem, so we find an equation relating f and θ. When you sketch the picture, you find

$$f(t) = 100 \cot \theta(t).$$

Differentiating,

$$f'(t) = -100 \frac{\theta'(t)}{\sin^2 \theta(t)}.$$

At $t = t_0$, we have

$$8 = -100 \frac{\theta'(t_0)}{\sin^2 \theta(t_0)}.$$

Again from the picture, you see $\sin \theta(t_0) = 100/200 = 1/2$. Therefore

$$8 = -100 \frac{\theta'(t_0)}{1/4}.$$

Solving for $\theta'(t_0)$, we get

$$\theta'(t_0) = -\frac{8}{400} = \boxed{-\frac{1}{50} \text{ rad/s}}.$$

(The angle is decreasing at the rate $\frac{1}{50}$ radian/sec.)

5. (20 points) A soda company wants to design a new cylindrical can. If the company wants to design a can with volume $16\pi\ cm^3$, and minimize how much aluminium it takes to make, what should the radius and the height of the can be, in cm?

Write r for the radius and h for the height.

We know $\pi r^2 h = 16\pi$, so $r^2 h = 16$, so $h = 16/r^2$. Note that by common sense, the radius and the height must be positive, so we are optimizing over the interval $r \in (0, \infty)$ We'd like to minimize the surface area of the cylinder, which is $2\pi rh + 2\pi r^2$, given the previous condition. Substituting in $16/r^2$ for h, we get the surface area is:

$$A(r) = 2\pi r^2 + \frac{32\pi}{r}.$$

The derivative is:

$$A'(r) = 2\pi(2r - \frac{16}{r^2}).$$

Setting this equal 0, we get:

$$2\pi(2r - \frac{16}{r^2}) = 0.$$

Dividing by 2π and multiplying by r^2 gives:

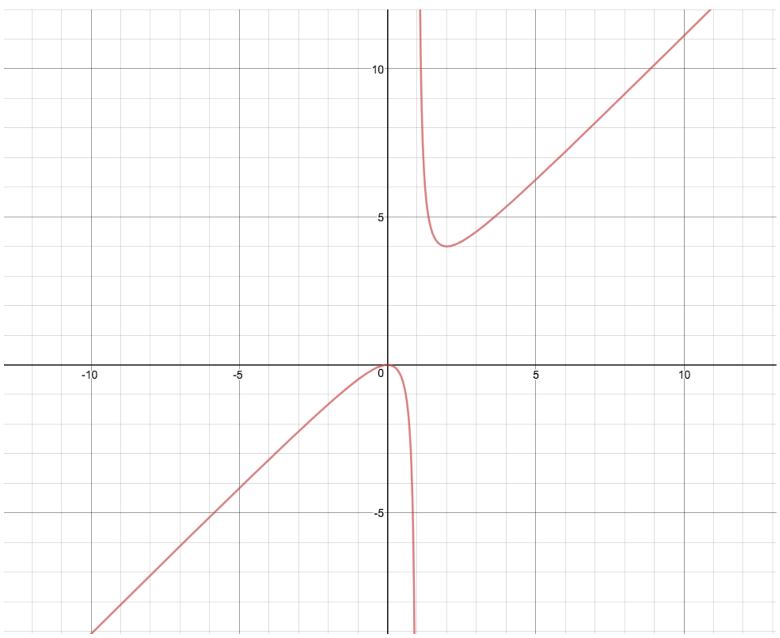
$$2r^3 = 16.$$

This happens at $r = \sqrt[3]{8} = 2$. Since this is the only critical number in the interval $(0, \infty)$, we can compute the sign of $A'(r)$ to see that $A'(r) < 0$ on $(0, 2)$ and $A'(r) > 0$ on $(2, \infty)$. By the I/D test we see that the function $A(r)$ has a unique absolute minimum at $r = 2$.

We can use the equation $h = 16/r^2$ to find that $h = 4$ when $r = 2$.

Answer: $\boxed{\text{radius 2 cm, height 4 cm}}$

6. (20 points) Sketch the curve $y = \dfrac{x^2}{x-1}$. Doing so, clearly indicate the following (if they exist): domain, x, y-intercepts, vertical/horizontal/slant asymptotes, intervals where the graph is increasing/decreasing/concave upward/concave downward, local maxima/minima.



Domain: $(-\infty, 1) \cup (1, \infty)$ (or $\{x \neq 1\}$)

x-intercept: 0, y-intercept: 0

VA is $x = 1$ as $\lim_{x \to 1^+} \dfrac{x^2}{x-1} = \infty$ (and can't have VA elsewhere because the function is continuous on its domain).

slant asymptote $y = x + 1$ because

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

thus

$$\lim_{x\to\infty}\left(\frac{x^2}{x-1}-(x+1)\right)=\lim_{x\to\infty}\frac{1}{x-1}=0.$$

(The same also holds for $x \to -\infty$.)

$$f'(x)=1-\frac{1}{(x-1)^2}$$

so $f'(x)=0$ exactly when $(x-1)^2=1$. This occurs for $x=0$ and $x=2$, giving critical numbers 2 and 4. Moreover $f'(x)<0$ exactly when $(x-1)^2<1$, namely $0<x<2$. Similarly $f'(x)>0$ when $x<0$ or $x>2$. By the I/D test, f is decreasing on $(0,2)$, increasing on $(-\infty,0)\cup(2,\infty)$.

$$f''(x)=\left(1-\frac{1}{(x-1)^2}\right)'=\frac{2}{(x-1)^3}.$$

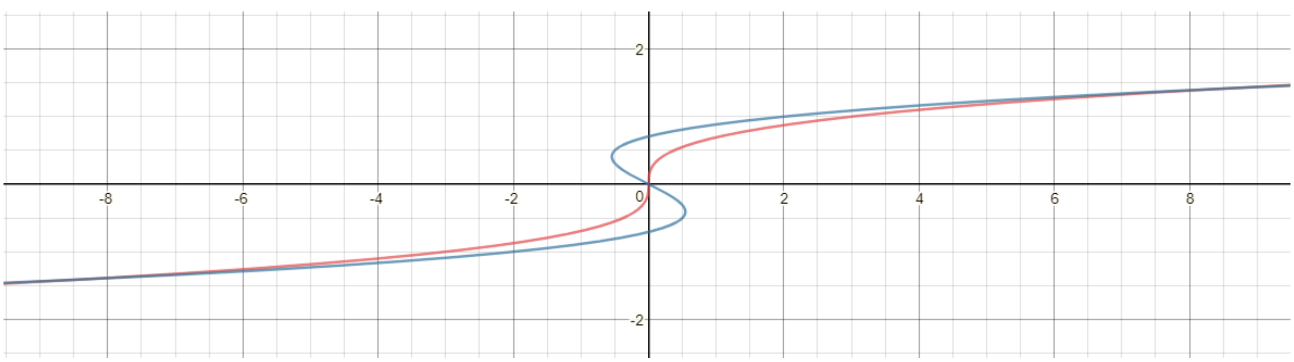
So $f''(x)>0$ on $x>1$ and $f''(x)<0$ on $x<1$. Hence the graph $y=f(x)$ is CU on $(1,\infty)$, CD on $(-\infty,1)$.

Finally by the first derivative test, f has a local max $f(0)=0$ at $x=0$ and a local min $f(2)=4$ at $x=2$.

7. (20 points) Find the total area enclosed by $x=3y^3$ and $x=4y^3-2y$.

First, find the points of intersection of these two curves by setting $3y^3=4y^3-2y$. Get $y(y^2-2)=0$. Solving, we get three points of intersection at $y=-\sqrt{2},0,\sqrt{2}$. Now, on the interval $(-\sqrt{2},0)$, $3y^3<4y^3-2y$ (you can plug in $y=-1$ to check). On the interval $(0,\sqrt{2})$, $4y^3-2y<3y^3$ (again, you can plug in $y=1$ to check).

When you plot the curve, you'll get



So the total area enclosed by the curves is

$$\int_{-\sqrt{2}}^{0}\left[(4y^3-2y)-(3y^3)\right]dy+\int_{0}^{\sqrt{2}}\left[(3y^3)-(4y^3-2y)\right]dy,$$

which is evaluated as $1+1=\boxed{2}$.

(Alternatively, since everything is symmetric about the origin, the area enclosed below the x-axis is the same as the area enclosed above the x-axis. This means that only one of the integrals needs to be evaluated, and then you can multiply the area it gives by 2.)

(Note: If it makes one psychologically more comfortable, one can swap x and y and compute the area enclosed by $y = 3x^3$ and $y = 4x^3 - 2x$. One arrives at the same formula as above, with x in place of y.)

8. (20 points) For any number x such that $-1 < x < 1$, show that $\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$.

Let's show that $f(x) = \sin^{-1} x + \cos^{-1} x$ is equal to $\pi/2$. Compute

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}}\right) = 0.$$

Since this is true on $(-1, 1)$, we see that f is a constant function. Computing the value at $x = 0$ (or any other value between -1 and 1),

$$f(0) = \sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

Therefore the constant function f must be equal to $\pi/2$.

(Note 1: This is similar to Example 6 of Section 4.2 or Exercise 35 of Section 4.2.)

(Note 2: One can use trigonometry to prove this without calculus.)

9. (20 points) Find the volume of the solid given by rotating the area bounded between $xy = 1$, $x = 1$, $x = 2$, $y = 0$ about the line $x = -1$.

Although it's not impossible to compute via cross sections, it's easier to use the cylindrical shell method. Then the volume is computed by the following integral:

$$\int_1^2 2\pi(x+1)\frac{1}{x}dx = \int_1^2 2\pi\left(1 + \frac{1}{x}\right)dx = 2\pi(x+\ln|x|)\Big]_1^2 = 2\pi(2+\ln 2) - 2\pi(1+0) = \boxed{2\pi(1+\ln 2)}.$$